Entropy2004, 6, 257-261

Entropy
ISSN 1099-4300
www.mdpi.org/entropy/

STATISTICAL CONVERGENT TOPOLOGICAL SEQUENCE ENTROPY MAPS OF THE CIRCLE

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Received: 26 August 2003 / Accepted: 17 December 2003 / Published: 19 March 2004

ABSTRACT: *A continuous map f of the interval is chaotic iff there is an increasing of nonnegative integers T such that the topological sequence entropy of f relative to T, $h_T(f)$, is positive [4]. On the other hand, for any increasing sequence of nonnegative integers T there is a chaotic map f of the interval such that $h_T(f)=0$ [7]. We prove that the same results hold for maps of the circle. We also prove some preliminary results concerning statistical convergent topological sequence entropy for maps of general compact metric spaces.*

Keywords: Statistical convergent, topological sequence, entropy, sequence entropy.

Mathematics Subject Classification (2000). Primary 26A18, 54h20.

INTRODUCTION

Let (X, ρ) be a compact metric space; denote by $C(X)$ the space of all maps of this space into itself. We will pay a special attention to the case when X is the circle $S = \{z \in C; | z | = 1\}$; the metric on S is given by $\| x, y \| = dist(\Pi^{-1}x, \Pi^{-1}y)$ where Π denotes the natural projection of the real line R onto S, i.e., $\Pi(x) = e^{2\pi ix}$. By N we denote the set of all positive integers.If $T = (t_i)_{i=1}^{\infty}$ is an arbitrary sequence of nonnegative integers then the (T,f,n)-trajectory of $x \in X$ is the sequence $(f^{t_i}x)_{i=1}^{\infty}$. The set of all periodic points of f is denoted by Per(f) and the set of periods of all periodic points of f by $P(f)$. A set $A \subseteq X$ is called a retract of X if there is a map $r : X \to A$ such that $r(a) = a$ for every $a \in A$.

Definition1: Let (X, ρ) be a compact metric space. The $(f^{t_i}x)_{i=1}^{\infty}$ is said to be statistical convergent to the $(f^{t_i}y)_{i=1}^{\infty}$, if for $\varepsilon > 0$, and for $x, y \in X$ such that

Definition2: Let (X, ρ) be a compact metric space. A map $f \in C(X)$ is said to be chaotic ifthere are points $x, y \in X$ such that

$$\limsup_{i\to\infty} \rho(f^i x, f^i y) > 0,$$

$$\liminf_{i\to\infty} \rho(f^i x, f^i y) = 0.$$

A set $A \subseteq X$ is said to be (T, f, ε, n)- statistical convergent separated if for any $x,y \in A$, $x \neq y$ there is an index i, $1 \le i \le n.$, such that $\rho(f^{t_i}x, f^{t_i}y) > \varepsilon$. Let st-Sep$(T, f, \varepsilon, n)$ denote the largest of cardinalities of all (T, f, ε, n) –statistical separated sets. Put

$$st - Sep(T, f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \tfrac{1}{n} st - Sep(T, f, \varepsilon, n).$$

A subset of *X* is said to be a *(T,f, ε ,n)*-st-span if it *(T,f, ε ,n)*st-spans X. Let st-Span*(T,f, ε ,n)* denote the smallest of cardinalities of all *(T, f, ε , n)*-st-spans. Put

$$st - Sep(T, f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \tfrac{1}{n} st - Span(T, f, \varepsilon, n).$$

Then st-Sep*(T,f)* = st-Span*(T,f)* we define the statistical convergent topological sequence entropy of *f* relative to *T*, $h_{st\text{-}T}(f)$, to be st-Sep*(T, f)*[3].

In [4] Franzová and Smítal, proved that a map *f* of the interval is chaotic if and only if there is an increasing sequence of nonnegative integers *T* such that $h_T(f) > 0$. A natural question arose whether there is some universal sequence which characterizes chaos. This is not the case as it was proved in [7] for any increasing sequence of nonnegative integers *T* there is a chaotic map *f* with $h_T(f) = 0$. The main aim of this paper is to prove the same results for statistical convergent topological sequence entropy maps of the circle.

Theorem1: A map $f \in C(S)$ is chaotic if and only if there is an statistical convergent sequence of nonnegative integers *T* such that $h_{st\text{-}T}(f) > 0$.

Theorem2: Let *X* be a compact metric space containing a homeomorphic image of an interval and let *T* be an statistical convergent sequence of nonnegative integers. Then there is a chaotic map $f \in C(X)$ such that $h_{st\text{-}T}(f) = 0$.

PRELIMINARY RESULTS

Let *(X, ρ) and (Y, σ)* be compact metric spaces, $f \in C(X)$; $g \in C(Y)$, and let $\pi : X \to Y$ be a map such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow{g} & Y \end{array}$$

commutes. In this situation we have the following.

Lemma1: Let *T* be an increasing sequence of nonnegative integers.
(i) if π is injective then $h_T(f) < h_T(g)$;
(ii) if π is subjective then $h_T(f) > h_T(g)$;
(iii) if π is bijective then $h_T(f) = h_T(g)$.

Proof:
(ii) and (iii). [5].
(i). We have that π is a homeomorphism between$X$ and πX. Thus, by (iii), $h_T(f) = h_T(g \mid \pi X)$. Now let $E \subseteq \pi X$ be $(T, g \mid \pi X, \varepsilon, n) - separated$. Trivially, it is also $(T, g, \varepsilon, n) - sparated$ which gives $h_T(g \mid \pi X) \le h_T(g)$.

Theorem3: Let *(X, ρ)* be a compact metric space, $f \in C(X)$, *T* be an statistical convergent sequence of nonnegative integers and *k* be a positive integer. Then there is an statistical convergent statistical sequence of nonnegative integers *S* such that $h_{st\text{-}S}(f^k) > h_{st\text{-}T}(f)$.

Proof: Since *X* is compact, $f, f^2, ..., f^{k-1}$ are equicontinuous, i. e., for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in X$ and $\rho(x; y) \le \delta$ then $\rho(f^{t_i} x, f^{t_i} y) < \varepsilon$ for $i = 1, ..., k-1$. We may assume that $\delta \le \varepsilon$.

Let $T=(t_i)_{i=1}^{\infty}$. Define $S=(s_i)_{i=1}^{\infty}$ as follows. Put $s_1=[\frac{t_1}{k}]$ (where [.] stands for the integer part) and for any m let s_{m+1} will be the first $[\frac{t_i}{k}]$ greater than s_m .

Let $E\subseteq X$ be an $(T,f,\varepsilon,n)-st-separeted$ set. We are going to show that E is a $(S,f^k,\delta,m)-st-separeted$ set where m is such that $s_m=[\frac{t_n}{k}]$.

To this end let $x,y\in E,\ \mathrm{x}\neq \mathrm{y}$. Then for some $i\in(\{1,2,...,n\},\ \rho(\mathrm{f}^{t_i}\mathrm{x},\mathrm{f}^{t_i}\mathrm{y})>\varepsilon$. Take j with $s_j=[\frac{t_i}{k}]$. Then $j\le m$ and from the definition of δ we have $\rho(\mathrm{f}^{k.s_i}\mathrm{x},\mathrm{f}^{k.s_i}\mathrm{y})>\delta$. Thus E is a $(S,f^k,\delta,m)-st-separeted$ set. From this we have $Sep(T,f,\varepsilon,n)\le Sep(S,f^k,\delta,m)$.

Now,

$$\begin{aligned}h_{st-T}(f)&=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\tfrac{1}{\mathrm{n}}\log(\text{st-Sep}(T,f^{\mathrm{k}},\varepsilon,n))\\&\le\lim_{\delta\to 0}\limsup_{\mathrm{n}\to\infty}\tfrac{1}{\mathrm{n}}\log(\text{st-Sep}(T,f^{\mathrm{k}},\delta,m))\\&\le\lim_{\delta\to 0}\limsup_{m\to\infty}\tfrac{1}{m}\log(\text{st-Sep}(T,f^{\mathrm{k}},\delta,m))=h_{\text{st-S}}(f^k).\end{aligned}$$

Corollary1: Let X be a compact metric space, $f\in C(X)$ and k be a positive integer. Then the following two conditions are equivalent:
(i) there is an increasing sequence T of nonnegative integers such that $h_T(\mathrm{f})>0$;
(ii) there is an increasing sequence T of nonnegative integers such that $h_T(\mathrm{f}^{\mathrm{k}})>0$.

In the sequel we will discuss the space of maps of the circle. The space $C(S)$ can be decomposed into the following classes[1], [10].

$C_1=\{f\in C(S);\ f$ has no periodic point$\}$;

$C_2=\{f\in C(S);\ P(f^n)=\{1\}$ or $P(f^n)0\{1,2,2^2,...\}$ for some $n\in N\}$;

$C_3=\{f\in C(S);\ P(f^n)=N$ for some $n\in N\}$.

According to this we will distinguish three different cases.

MAPS WITHOUT PERIODIC POINTS

Throughout this section we assume $f\in C(S)$ to have no periodic point. We are going to show that Theorem1 holds for such continuous maps. Since, by [9], f is not chaotic, we need only to show that $h_T(f)=0$ for any increasing sequence T. So fix T. If f is a homeomorphism then $h_T(f)=0$. Otherwise, there is a nowhere dense perfect set E which is the only ω-limit set of f, all (closed) contiguous intervals are wandering, the preimage of any contiguous interval is a contiguous interval, the image of any contiguous interval is either a contiguous interval or a point from E. Moreover, $f|_E$ is monotone. By linear extension of $f|_E$ we obtain a monotone map $g\in C(S)$. By [8], $h_T(g)=0$ By Lemma1(i), $h_T(f|_E)\le h_T(g)$.

Hence,

$$\limsup_{\mathrm{n}\to\infty}\tfrac{1}{\mathrm{n}}\log\mathrm{Span}(T,f|_E,\varepsilon,\mathrm{n})=0\ \text{ for any }\ \varepsilon>0.$$

Now fix an arbitrary $\varepsilon>0$. We are going to estimate st- Span(T,f,ε,n). Let $I_1,...,I_k$ be all contiguous intervals longer than $\frac{\varepsilon}{2}$. Let A be a $(T,f|_E,\frac{\varepsilon}{2},n)$ st-span. Take any point x whose *(T,f, n)*-trajectory lies in $S\setminus\bigcup_{i=1}^{k}I_i$. If $\mathrm{x}\in\mathrm{E}$ then x is *(T, f,* ε *, n)*st-spanned by A.

For $x \notin E$ put y to be an endpoint of the contiguous interval which contains x. Then,

$$\| f^{t_i} x, f^{t_i} y \| \leq \tfrac{\varepsilon}{2} \text{ for all } 1 \leq i \leq n.$$

Since $y \in E$ is $(T, f, \tfrac{\varepsilon}{2}, n)st - spanned$ by a point $z \in A$, the set A obviously (T, f, ε, n) st-spans all such points x. So it remains to consider those points whose *(T, f, n)*-trajectories meet $\bigcup_{i=1}^{k} I_i$. Fix $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. We are going to show that there is a set of cardinality at most $n.k.N^k$ which (T, f, ε, n) st-spans all considered points. It is sufficient to show that there is a set with cardinality at most N^k which (T, f, ε, n) st-spans the set $I(t_i, I_j) = \{x \in S; f^{t_i} x \in I_j\}$ (for fixed $1 \leq j \leq n$ and $1 \leq j \leq k$). First, it is obvious that $I(t_i, I_j)$ is a contiguous interval. Consider its *(T, f, n)*-trajectory $(f^{t_1} I(t_i, I_j), \ldots, f^{t_n} I(t_i, I_j))$. Each element in this trajectory is either a contiguous interval or a point from *E*. At most *k* of them have lengths greater than or equal to ε - cut each of such elements to *N* segments shorter than ε. All the other elements of the trajectory will be considered to be segments themselves. To each $x \in I(t_i I_j)$ assign its code- the sequence $(S_1(x), \ldots, S_n(x))$ where $S_\ell(x)$ is the segment containing $f^{t_\ell} x$. We have at most N^k different codes and all points with the same code can be *(T, f, ε , n)*-st-spanned by one point. From what has been said above we see that

$$st - Span(T, f, \varepsilon, n) \leq st - Span(T, f|_E, \tfrac{\varepsilon}{2}, n) + n.k.N^k$$

which finishes the proof of Theorem1 for maps without periodic points.

MAPS WITH PERIODIC POINTS

We will first deal with the case C_2. We know that for any $n \in N$ f is chaotic if and only if f^n is chaotic. Taking into account Corollary1 we can without loss of generality assume that $P(f) = \{1\}$ or $P(f) = \{1,2,,2^2,\ldots\}$. Since *f* has a fixed point, by [10] there is a lifting *F* and an *F*-invariant compact interval *J* longer than 1. In the following discussion of the case C_2 we will write *F* and Π instead of $F|_J$ and $\Pi|_J$, respectively, as in the next commutative diagram

$$\begin{array}{ccc} J & \xrightarrow{F} & J \\ \pi \downarrow & & \downarrow \pi \\ S & \xrightarrow{f} & S \end{array}$$

Note that if $x,y \in J$ then $\| \Pi x, \Pi y \| \leq |x\text{-}y|$ with the equality whenever $|x\text{-}y| \leq \frac{1}{2}$.

Lemma2: *F* is chaotic if and only if *f* is chaotic[8],[10].

Lemma3: Let *F* be chaotic. Then there is an statistical convergent sequence *T* such that $h_{st\text{-}T}(f) > 0$.

Proof: If *F* has a periodic point of period $k.2^m$ where $k>1$ is odd then, by Sharkovsky theorem, it has also a periodic point of period $k'.2^m$ where $k>0$ diam $J + 1$ is odd. Since $\Pi|_J$ is to most [diam *J*] + 1 to one, *f* has a periodic point of period $k''.2^{m'}$ where $k'' > 1$ is odd. This is a contradiction since *P(f)* is $\{1\}$ or $\{1,2,,2^2,\ldots\}$. So *F* is of type 2^∞, chaotic. By [10] there is an orbit of periodic intervals of period $p>$diam *J* such that F^p is chaotic on each of them. At least one interval *K* in this orbit is shorter than 1. Then $\Pi|_K$ is injective and so $F^p|_K$ is topologically conjugate with $f^p|_{\Pi K}$. There is an statistical convergent sequence of nonnegative integers S such that $h_{st\text{-}S}(F^p|_K) > 0$. Since $h_{p.st\text{-}S}(f) = h_{st\text{-}S}(f^p)$ it is sufficient to use Lemma1(iii) and (i) to get

$$h_{p.st-S}(f) \geq h_{st-S}(f^p|_{\Pi K}) = h_{st-S}(F^p|_K). \Diamond$$

Proof of Theorem1: We are going to show that Theorem 1 holds for maps from the class C_2. Let $f \in C_2$ be chaotic. Then we obtain the required result using Lemma 1 and Lemma2.

Now let $f \in C_2$ and let there be an statistical convergent sequence of nonnegative integers T such that $h_{st\text{-}T}(f)>0$. Lemma(ii) implies that $h_{st\text{-}T}(F)>0$ where F has the same meaning as above. F is chaotic.

Finally we will discuss the situation for maps from the remaining class C_3. So let $P_{st\text{-}}(f^n)=N$ for some n. By [2] We have that $h_{st\text{-}}(f^n)$ is positive and so is $h_{st\text{-}}(f)$. Then we have that $f^{m.n}$ is strictly turbulent for a suitable $m \in N$ which implies that f is chaotic for the same reason as in the interval case. This finishes the proof of Theorem1.

Proof of Theorem2: The space X contains a homeomorphic image J of the interval [0,1]. The set J is a retract of X by [6]. Let $r : X \to J$ be a corresponding retraction. By [7] there is a chaotic onto map $g \in C([0,1])$ such that $h_{st\text{-}T}(g) = 0$. Let $\tilde{g} \in C(J)$ be a map topologically conjugate with g. Define $f \in C(X)$ by $f = \tilde{g} o r$. Since $\bigcap_{i=0}^{\infty} f^i X = fX = J$, we have that $h_{st\text{-}T}(f) = h_{st\text{-}T}(f|_J) = 0$.

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