From classical statistical model to quantum model through ignorance of information

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Abstract: We consider transition from the classical statistical model to the quantum statistical model through ignorance (of huge volume) of information in process of construction of a wave function – a complex probability amplitude. Our approach clarifies relation between classical and quantum statistical models (and hence relation between classical and quantum information theories). In particular, it can be considered as a step toward demystification of quantum theory. The notion of context (complex of physical conditions) is basic in this paper. We show that the main structures of quantum theory (interference of probabilities, Born's rule, complex probabilistic amplitudes, Hilbert state space, representation of observables by operators) are present in a latent form in the classical Kolmogorov probability model. However, this model should be considered as a calculus of contextual probabilities.In our approach it is forbidden to consider abstract context independent probabilities: "first context and then probability." In this way we obtain interference of probabilities without to appeal to the Hilbert space formalism or wave mechanics.
Our formalism can be applied to various domains outside quantum physics, e.g., cognitive sciences, psychology, economics, chemistry: roughly speaking in any domain in that it is impossible to extract the whole information about an ontic (or to say realistic) model and hence the epistemological (experimental or to say observational) model is constructed through huge reduction of information volume. We introduced a notion of *prespace* – space of states containing complete information about systems under consideration. Classical phase space and quantum Hilbert space are obtained as projections of such a prespace.

Keywords: classical and quantum statistical model; construction of wave function; reduction of information; applications in cognitive sciences, psychology, economics, chemistry

1 Introduction

1.1 The orthodox Copenhagen interpretation of quantum mechanics

It is well known that some creators of quantum mechanics (e.g. Nils Bohr) believed that quantum mechanics provides the complete (final) description of physical reality; that it would be impossible to create a deeper theory providing more detailed description of physical phenomena. This viewpoint was the basis of so called *orthodox Copenhagen interpretation of quantum mechanics.* It is also well known that quantum mechanics (at least for this interpretation) is an extremely mysterious theory inducing numerous paradoxes. Adherents of the orthodox Copenhagen interpretation teaches us that one could not escape quantum mysteries or explain them in classical terms.

In particular, there is commonly believed that *quantum randomness* differs crucially from classical ensemble randomness which was used as the probabilistic basis of the classical statistical mechanics. In opposite to classical randomness, quantum randomness is *irreducible*, see e.g. J. von Neumann [1]. It could not be reduced to randomness based on variability of (objective) properties of systems in an ensemble (which was prepared under the same complex of physical conditions – physical *context C*). J. von Neumann [1] emphasized that quantum randomness is individual randomness, i.e., it is associated with an individual quantum system. However, he understood well that it is in principle impossible to verify individual randomness, because we could not perform a series of measurements on a fixed quantum particle without to disturb (or destroy) it. Even in quantum mechanics (as well as in classical statistical mechanics) probabilistic predictions can be verified only via observations for large ensembles of particles prepared under the same *context.* Nevertheless, J. von Neumann [1] was sure that the ensemble randomness of quantum particles is just an exhibition of individual randomness; the main distinguishing features of quantum probability could not be explained with the aid of ensemble representation. This point of view to quantum randomness became dominating in quantum community.

1.2 Interference, wave-particle duality

In particular, there is a rather common opinion that *interference of probabilities* exhibited in the famous two slit experiment (and other experiments on the wave-like features of quantum systems) could be explained only with the aid of individual quantum randomness. Interference is understood as a kind of *self-interference.* Such a viewpoint to quantum interference was the basis of so called wave-particle dualism and *Bohr's principle of complementarity.* By orthodox Copenhagen interpretation it is forbidden to consider such properties of a quantum system as its position and momentum; quantum particles do not have trajectories. We recall that this Copenhagen story began long before the modern Bell's story about incompatibility of local realism and quantum formalism.

1.3 Incompleteness as the essence of quantum description

In this paper we present totally different view to quantum mechanics. We show that quantum mechanics is not complete theory at all. Moreover, incompleteness is the essence of quantum theory. This theory is nothing else as a very special way of description of systems (e.g. physical or biological) in situations in which it is impossible to obtain complete information about systems. Suppose that the detailed description of some class of systems can be given through knowledge of

distributions of a set of random variables RV. Suppose that by some reasons (e.g. technological or social, or economical, or political) we are not able to perform measurements of the whole collection of random variables $\xi \in RV$. Thus we are not able to obtain the complete statistical description of systems under consideration. Nevertheless, we can try to create a probabilistic model which would provide incomplete description of systems on the basis of statistics available from measurements which we can perform. We shall show that quantum mechanics can be interpreted as theory of such a type. In our approach quantum mechanics is a probabilistic description of physical reality based on the possibility to measure one special chosen ("fundamental") physical variable – *position* and its conjugate variable – *momentum.*

Such an approach could be applied not only to probabilistic description of microsystems, but to any class of systems (physical, biological, social,...) for which it is impossible to obtain the complete set of statistical data. Only special class of random variables can be measured. Corresponding probabilities are used for computation of complex probability amplitudes – quantum-like states.

On one hand, in this paper we show that quantum probabilistic description is just the result of ignorance of huge set of statistical data about microsystems. In such an approach quantum mechanics is *neither complete nor mysterious.* On the other hand, we show that our quantum-like formalism – the probabilistic representation based on ignorance of information – can be used in cognitive and social sciences and general theory of complex information systems. We describe functioning of the brain as *quantum-like* computer – computer performing computations by using very rough images (encoded by complex probability amplitudes) of neuronal activity. We even speculate that such a functioning based on specially chosen (through two "fundamental observables") projections of neuronal world may have some relation to conscious processing of information. May be the latter contribution is even more interesting than the representation of quantum mechanics as a projection of classical statistical model.

1.4 Combining neuronal realism with quantum formalism

The main distinguishing feature of our quantum-like approach to cognitive sciences is the possibility to combine neuronal realism with mathematical formalism of quantum mechanics. In our model "quantum probabilistic waves" (represented in the mathematical model by complex probability amplitudes) are produced by ensembles of neurons. There is nothing mysterious in wave-like dynamics of mental information. Such a dynamics (which we use to simulate the process of thinking) is the result of the ability of brain to perform a quantum-like projection of the ocean of neuronal information. At each instant of (mental) time the brain selects two fundamental variables (selects representation of the neuronal ocean[1]) and creates the image of activity of neuronal ocean given by a complex probability amplitude (see section 4 for the algorithm of producing a complex probability amplitude from statistical data).[2] Our fundamental conjecture is that the brain operates (at least on the highest level of mental functioning) with such quantum-like images by using algorithms of quantum computing. Thus one can call the brain *quantum-like computer.* Its functioning is mathematically described by the conventional theory of quantum computing, but physically it has nothing to do with the conventional quantum computer. There is no need in such still mysterious things as *quantum parallelism* or superposition of states for an individual

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[1]Cf. with "Solaris" of Stanislav Lem and especially with the corresponding film of Andrei Tarkovsky.

[2]Of course, it is assumed that the brain is able to collect this data. This collecting could not be performed instantaneously. Therefore we speak about moments of mental time which correspond to intervals of physical time.

system. Escaping from these mysteries is especially important for neurology. In principle, one may speculate as long as he likes about superposition of states for an individual electron – it is impossible (at least directly) to test this assumption (see our discussion on quantum randomness). But it is really impossible to do this with an individual neuron. As was correctly remarked by Roger Penrose, a neuron could not be at the same time in the state of firing and nonfiring.

We can speculate that even collective cognitive systems (human societies, states, nations, groups of animals, birds, insects) are able to create quantum-like probabilistic representations of information. One could say that such cognitive systems are driven by probabilistic quantum-like waves. Finally, we remark that one could not exclude that such representations could be created by nonliving complex information systems. Our approach opened the way to *quantum-like artificial intelligence.*

1.5   NO-GO theorems

Since the first days of creation of quantum mechanics, physicists, mathematicians and philosophers are involved in stormy debates on the possibility to create a classical prequantum statistical model, see for example [1], [2] (and recent publications [3]–[8]). Here "classical statistical" has the meaning of a realistic model in that physical variables can be considered as objective properties and probabilities can be described by the classical (Kolmogorov) measure-theoretic model. There is a rather common opinion that it is impossible to construct such a prequantum model. Such an opinion is a consequence of Bohr's belief that quantum mechanics is a *complete theory.* Therefore it is in principle impossible to create a deeper description of physical reality. In particular, there is a rather common belief that quantum randomness is irreducible, see e.g. von Neumann [2] (in the opposite to classical randomness which is reducible in the sense that it can be reduced to ensemble randomness of objective properties). There is a huge activity in proving various mathematical "NO-GO" theorems (e.g. von Neumann, Kochen-Specker, Bell,...). Many people think that with the aid of such mathematical exercises it is possible to prove completeness of quantum mechanics. As was pointed out in the preface to the conference proceedings [6], such an approach can not be justified, because we do not know the *correspondence rules* between prequantum and quantum models (since we do not have yet any prequantum realistic statistical model). J. von Neumann presented in his book [2] the list of his beliefs about features of such a prequantum→quantum map J. Later this list was strongly criticized by many authors (including J. Bell). In particular, there was criticized the assumption on on-to-one correspondence between the set of classical prequantum physical variables V and the set of quantum observables O. There was also pointed out that von Neumann assumption that $J(a + b) = J(a) + J(b)$ for any two physical variables (so without the assumption that observables $J(a)$ and $J(b)$ can be measured simultaneously) is nonphysical. Then different authors proposed their own lists of beliefs about features of the map J which (as they think) are natural. These lists (including Bell's list) were again criticized. Such a "NO-GO" activity and its critique can be continued as long as we want. In [7]-[9] I proposed to start the activity in the opposite direction. Instead of looking for lists of assumptions on the prequantum→quantum map J which would imply a new "NO-GO" theorem, it seems to be more natural to try to find such lists of features of J which would give the possibility to create a natural prequantum model. My solution of this problem is very close to ideas of L. De Broglie and D. Bohm who thought that not all physical variables have "equal rights." They thought that the position variable plays a special role. This is a fundamental physical variable and the corresponding observable is also fundamental; De Broglie pointed out that all measurements in physics could be (at least in principle) reduced to position measurements. I also think so.

In [7]–[9] there was show that quantum mechanics can be considered as an image of a prequantum classical statistical model if the list of features of the correspondence map J contains just one postulate:

Postulate RO. (Reference Observables) *There exist two fundamental physical observables $J(a)$ and $J(b)$ which correspond to prequantum physical variables a and b. We call them reference observables.*

2   Information prespace

Denote the set of fundamental parameters of nature (if you like hidden variables) by symbol Ω. We call it *prespace*. Denote by $V(\Omega)$ the set of physical variables. In a mathematical model $V(\Omega)$ is realized as some class of functions $d : \Omega \to \mathbf{R}$, where $\mathbf{R}$ is the set of real numbers. We develop the most general abstract approach. We do not make any assumption on algebraic or topological structure of prespace. We will not study dynamics of parameters $\omega = \omega(t)$ in Ω; for our general probabilistic considerations we need not pay attention to algebraic and topological features of prespace. In principle, Ω might be a manifold of huge (may be even infinite) dimension; it need not be a manifold over the field of real numbers; it might be a non-Archimedean (e.g. p-adic) manifold, cf. [8].

It is supposed that prespace is endowed with the structure of the Kolmogorov probability space: $(\Omega, F, \mathbf{P})$. Here F is a σ-algebra of subsets of Ω and $\mathbf{P}$ is a probability measure on F. Typically elements of F are interpreted as *events*, for example, in the conventional Kolmogorov model. We propose *contextual interpretation* of the Kolmogorov probability space. By a *physical context* we understand a complex of physical conditions, e.g. experimental conditions (but we do not assume that any context can be realized experimentally; contexts belong to *ontic models*, see [6]). Contexts are represented by elements of F. So we use the set-theoretic description of complexes of physical conditions. In principle, we can choose the set of random variables (measurable functions) as the set physical variables $V(\Omega)$.

In the conventional model the conditional probability is mathematically defined by the Bayes' formula: $\mathbf{P}(A/C) = \mathbf{P}(AC)/\mathbf{P}(C), \mathbf{P}(C) \neq 0$. In our model we do not have events, we consider contextual probability:

$$\mathbf{P}(d = x/C) = P(\omega \in C : d(\omega) = x)/P(C).$$

This is the probability that the variable d is equal to x under the complex of physical conditions C.

We gave the description of ontic model of physical reality: reality as it is. We are now going to consider epistemic ("observational") models of reality, see [3] for details. There are two well known epistemic models:

a). Ordinary classical mechanics on the physical phase space $\mathbf{R}^3 \times \mathbf{R}^3$.

b). Quantum mechanics on the complex Hilbert space H.

We emphasize that a classical model on prespace Ω should not be identified with the ordinary classical model on $\mathbf{R}^3 \times \mathbf{R}^3$. These are two different levels of description of nature. In fact, the ordinary classical model on $\mathbf{R}^3 \times \mathbf{R}^3$ (as well as the quantum model) is epistemic for our ontic prespace model, see section 10.

We now suppose that Postulate RO holds for the correspondence between prequantum classical statistical model and quantum statistical model. Denote such reference physical variables a and b; they belong to the space $V(\Omega)$.

3   Interference

Let $a = a_1, ..., a_n$ and $b = b_1, ..., b_n$ be discrete random variables and let $C \in F$. Then the classical *formula of total probability* holds:

$$\mathbf{P}(b = b_i/C) = \sum_n \mathbf{P}(a = a_n/C)\mathbf{P}(b = b_i/a = a_n, C).$$

This formula is well know in statistics. This is the basic formula of Bayesian analysis.

Let a, b be two random variables. They are said to be *supplementary* if $\mathbf{P}(b = x, a = y) \neq 0$ for all their values x and y. We invented a new term "supplementary". In principle, it would be natural to use the term "complementarity". Unfortunately, this term was already reserved by N. Bohr who used it to express *mutual exclusivity*. In our case physical variables a and b are not mutually exclusive, they are well defined for any $\omega \in \Omega$. Sometimes in quantum mechanics there is used term "incompatible". Incompatibility is impossibility of simultaneous measurement. Such a notion is totally meaningless inside the ontic model.

We shall consider the case of *supplementary dichotomous random variables* $a = a_1, a_2, b = b_1, b_2$. We set $Y = \{a_1, a_2\}, X = \{b_1, b_2\}$ ("spectra" of random variables a and b). We set

$$C_y = \{\omega \in \Omega : a(\omega) = y\}, y \in Y.$$

These sets represent contexts corresponding to selections with respect to fixed values $a = y$. By Postulate RO we can identify reference varaiables a and b with corresponding observables. Therefore the contexts represented by sets C_y are experimentally realizable. In [8]–[9] there was proved the following interference formula of total probability:

$$\mathbf{P}(b = x/C) = \sum_{j=1}^2 \mathbf{P}(a = a_j/C)\mathbf{P}(b = x/a = a_j) + 2\lambda(b = x/a, C)\sqrt{\prod_{j=1}^2 \mathbf{P}(a = a_j/C)\mathbf{P}(b = x/a = a_j)},$$

where

$$\lambda(b = x/a, C) = \frac{\mathbf{P}(b = x/C) - \sum_{j=1}^2 \mathbf{P}(b = x/a = a_j)\mathbf{P}(a = a_j/C)}{2\sqrt{\prod_{j=1}^2 \mathbf{P}(a = a_j/C)\mathbf{P}(b = x/a = a_j)}}. \tag{1}$$

In fact, this formula is just a representation of the probability $\mathbf{P}(b = x/C)$ in a special way. The $\lambda(x/a, C)$ were called the *coefficients of supplemetarity.*
Suppose that, for every $x \in X$,

$$|\lambda(b = x/a, C)| \leq 1 .$$

In this case we can introduce new statistical parameters $\theta(b = x/a, C) \in [0, 2\pi]$ and represent the coefficients of statistical disturbance in the trigonometric form:

$$\lambda(b = x/a, C) = \cos\theta(b = x/a, C).$$

Parameters $\theta(b = x/a, C)$ are called *probabilistic phases* ("angles of supplemetarity"). We remark that in general there is no geometry behind these phases. By using the trigonometric representation of the coefficients λ we obtain the well known *formula of interference of probabilities* which is typically derived by using the Hilbert space formalism.

If both coefficients λ are larger than one, we can represent them as

$$\lambda(b = x/a, C) = \pm \cosh \theta(b = x/a, C)$$

and obtain the formula of hyperbolic interference of probabilities; there can also be found models with the mixed hyper-trigonometric behavior, see [7]–[9].

4   Derivation of complex probabilistic amplitudes from statistical data

We recall that we consider the case of supplementary dichotomous random variables $a = a_1, a_2, b = b_1, b_2$. This pair of variables will be fixed. We call such variables **reference variables.** For each pair a, b of reference variables we construct a representation of the contextual Kolmogorov model in the Hilbert space ("quantum-like representation"). We start with the probabilistic representation of trigonometric contexts:

$$C^{\mathrm{tr}} = \{C : |\lambda(x/a, C)| \le 1, x \in X\}.$$

Of course, the system C^{tr} depends on the choice of a pair of reference observables, $C^{\mathrm{tr}} \equiv C^{\mathrm{tr}}_{b/a}$. We set

$$p_C^a(y) = \mathbf{P}(a = y/C), p_C^b(x) = \mathbf{P}(b = x/C), p(x/y) = \mathbf{P}(b = x/a = y), x \in X, y \in Y.$$

Let context $C \in C^{\mathrm{tr}}$. The interference formula of total probability can be written in the following form:

$$p_c^b(x) = \sum_{y \in Y} p_C^a(y) p(x/y) + 2 \cos \theta_C(x) \sqrt{\Pi_{y \in Y} p_C^a(y) p(x/y)},$$

where

$$\theta_C(x) = \theta(b = x/a, C) = \pm \arccos \lambda(b = x/a, C), x \in X.$$

By using the elementary formula:

$$D = A + B + 2\sqrt{AB} \cos \theta = |\sqrt{A} + e^{i\theta}\sqrt{B}|^2,$$

for $A, B > 0, \theta \in [0, 2\pi]$, we can represent the probability $p_C^b(x)$ as the square of the complex amplitude (Born's rule):

$$p_C^b(x) = |\varphi_C(x)|^2, \tag{2}$$

where a complex probability amplitude is defined by

$$\psi(x) \equiv \varphi_C(x) = \sqrt{p_C^a(a_1) p(x/a_1)} + e^{i\theta_C(x)} \sqrt{p_C^a(a_2) p(x/a_2)}. \tag{3}$$

We denote the space of functions: $\psi : X \to \mathbf{C}$ by the symbol $\Phi = \Phi(X, \mathbf{C})$. Since $X = \{b_1, b_2\}$, the Φ is the two dimensional complex linear space. By using the representation (3) we construct the map $J^{b/a} : C^{\mathrm{tr}} \to \Phi(X, \mathbf{C})$ which maps contexts (complexes of, e.g., physical conditions) into complex amplitudes. The representation (2) of probability is nothing other than the famous **Born rule.** The complex amplitude $\psi_C(x)$ can be called a **wave function** of the complex of physical

conditions (context) C or a (pure) *state*. We set $e_x^b(\cdot) = \delta(x - \cdot)$. The Born's rule for complex amplitudes (2) can be rewritten in the following form:

$$p_C^b(x) = |(\psi_C, e_x^b)|^2, \tag{4}$$

where the scalar product in the space $\Phi(X, C)$ is defined by the standard formula:

$$(\psi_1, \psi_2) = \sum_{x \in X} \psi_1(x)\bar{\psi}_2(x). \tag{5}$$

The system of functions $\{e_x^b\}_{x \in X}$ is an orthonormal basis in the Hilbert space $H = (\Phi, (\cdot, \cdot))$. By using the Hilbert space representation of the Born's rule we obtain the Hilbert space representation of the expectation of the (Kolmogorovian) random variable b:

$$E(b/C) = \sum_{x \in X} x|\psi_C(x)|^2 = (\hat{b}\psi_C, \psi_C),$$

where the (self-adjoint) operator $\hat{b} : H \to H$ is determined by its eigenvectors: $\hat{b}e_x^b = xe_x^b, x \in X$. This is the multiplication operator in the space of complex functions $\Phi(X, \mathbf{C}) : \hat{b}\psi(x) = x\psi(x)$. It is natural to represent this random variable (in the Hilbert space model) by the operator $\hat{b}$. We have Born's rule not only for the b-variable, but also for the a-variable:

$$p_C^a(y) = |(\psi_C, e_y^a)|^2, y \in Y.$$

The basis $\{e_y^a\}$ corresponding to the a-observable is defined in [9]. We remark that operators $\hat{a}$ and $\hat{b}$ representing the reference observables $J(a)$ and $J(b)$ do not commute, see [9] (this is a consequence of supplementarity of reference variables a and b). We can also consider hyperbolic contexts:

$$C^{\mathrm{hyp}} = \{C : |\lambda(x/a, C)| \geq 1, x \in X\}.$$

Such contexts (complexes of physical conditions) are represented by amplitudes ψ_C taking values in the algebra of so called hyperbolic numbers, see [7]–[9] for details.

5   Quantum-like mind

We consider examples of *cognitive contexts*:

1). C can be some selection procedure which is used to select a special group S_C of people or animals. Such a context is represented by this group S_C (so this is an ensemble of cognitive systems).

2). C can be a collection of painting, C_{painting}, (e.g. the collection of Hermitage in Sankt-Peterburg) and people interact with C_{painting} by looking at pictures(and then there are asked questions about this collection to those people).

3). C can be, for example, "context of classical music", $C_{\mathrm{cl.mus.}}$, and people interact with $C_{\mathrm{cl.mus.}}$ by listening in to this music. In principle, we need not use an ensemble of different people. It can be one person whom we ask questions each time after he has listened in to CD (or radio) with classical music. In the latter case we should use not ensemble, but frequency (von Mises) definition of probability.

The last example is an important illustration why from the beginning we prefer to start with the general contextualist ideology and only then we consider the possibility to represent contexts by ensembles of systems. A cognitive context should not be identified with an ensemble of cognitive systems representing this context. For us $C_{\text{cl.mus.}}$ is by itself an element of reality. We can also consider *social contexts:* proletariat-context, bourgeois-context; or war-context, revolution-context,.... Thus our model can be used in social and political sciences (and even in history). We can try to find quantum-like statistical data in these sciences.

We describe *mental interference experiment*:

Let $b = x_1, x_2$ and $a = y_1, y_2$ be two dichotomous mental observables: x_1='yes', x_2='no', y_1='yes', y_2='no'. Observables can be two different questions or two different types of cognitive tasks. We use these two fixed reference observables for probabilistic representation of cognitive contextual reality given by C.

We perform observations of a under the complex of cognitive conditions C :

$$p^b(x) = \frac{\text{the number of results } b = x}{\text{the total number of observations}}, \quad x \in X.$$

So $p^b(x)$ is the probability to get the result x for observation of the b under the complex of cognitive conditions C. In the same way we find probabilities $p^a(y)$ for the a-observation under the same cognitive context C.

We suppose, there can be created cognitive contexts C_y corresponding to selections with respect to fixed values of the a-observable. The context C_y (for fixed $y \in Y$) can be characterized in the following way. By measuring the a-observable under the cognitive context C_y we shall obtain the answer $a = y$ with probability one. We perform now the b-measurements under cognitive contexts C_y for $y = y_1, y_2$, and find the probabilities:

$$p(x/y) = \frac{\text{the number of the result } b = x \text{ under context } C_y}{\text{the total number of observations under context } C_y},$$

$x \in X, y \in Y$. For example, by using the ensemble approach to probability we have that the probability $p(x_1/y_2)$ is obtained as the frequency of the answer $b = x_1 = $ 'yes' in the ensemble of cognitive system that have already answered $a = y_2 = $ 'no'. Thus we first select a subensemble of cognitive systems who replies 'no' to the a-question: $C_{a=no}$. Then we ask systems belonging to $C_{a=no}$ the b-question.

In the quantum-like statistical test for a cognitive context C we calculate the coefficient of supplementarity $\lambda(b = x/a, C)$, see (1). An empirical situation with $\lambda(b = x/a, C) \neq 0$ would yield evidence for quantum-like behaviour of cognitive systems. In this case, starting with (experimentally calculated) coefficient of supplementarity $\lambda(b = x/a, C)$ we can proceed either to the conventional Hilbert space formalism (if this coefficient is bounded by 1) or to the hyperbolic Hilbert space formalism (if this coefficient is larger than 1).

Complex probability amplitude $\psi = \psi_C$ representing a cognitive context C we call *mental wave function.* This is nothing else than a special mathematical encoding of probabilistic information about this context which can be obtained with the aid of mental reference observables a and b.

6   Quantum-like representation of neuronal structures

We emphasize that the quantum-like representation is created through a projection of underlying mental realistic model to the complex Hilbert space. Such a projection induces a huge loss of

information about the underlying mental model. Thus the quantum-like model gives a very rough image of the realistic model.

Let us consider two coupled neural networks N_1 and N_2. We assume that they are strictly hierarchic in the sense that there are "grandmother" neurons n_1 and n_2 in networks N_1 and N_2, respectively. The integral network $N = N_1 + N_2$ interacts with contexts C which are given by input signals into both networks. For example, contexts $\mathcal{C} = \{C\}$ can be visual images and the integral network N recognizes those images (e.g. N_1 is responsible for countors and N_2 for colors). We use so called frequency-domain approach and assume that cognitive information is presented by frequencies of firing of neurons. Consider two reference observables a, b where $a = 1 : n_1$ firing, and $a = 0 : n_1-$nonfiring, and $b = 1 : n_2-$firing, and $b = 0 : n_2-$nonfiring. Our quantum-like formalism gives the possibility to represent each context C (e.g., an image C) by a complex probability amplitude ψ_C. Here probabilities $\mathbf{P}(b = x/C), \mathbf{P}(a = y/C)$ are defined as frequencies. Such an amplitude can be reconstructed on the basis of measurements on grandmother neurons n_1 and n_2. Of course, ψ_C gives only a projection of the neuronal image of the context C. The complete neuronal image is given by frequencies of firing of all neurons in the network N and the QL-image ψ_C is based only on frequencies of firings of grandmother neurons. However, we could not exclude that cognition (and consciousness) is really based on such a QL-projecting of neronal states, see section 7.

7   Quantum-like consciousness

The brain is a huge information systems which contains millions of minds. It could not "recognize" (or "feel") all those minds at each instant of time t. Our fundamental hypothesis is that the brain is able to create the QL-representations of minds. At each instant of time t the brain creates the QL-representation of its mental context C based on two supplementary mental (self-)observables a and b. Here $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ can be very long vectors of nonsupplementary dichotomous observables. The (self-)reference observables can be chosen (by the brain) in different ways at different instances of time. Such a change of the reference observables is known in cognitive sciences as a *change of representation.*

A mental context C in the $b/a-$ representation is described by the mental wave function ψ_C. We can speculate that the brain has the ability to feel this mental field as a distribution on the space X. This distribution is given by the norm-squared of the mental wave function: $|\psi_C(x)|^2$. This mental QL-wave contributes into the deterministic dynamics of minds, e.g. by inducing Bohmian quantum potential, see e.g. [10], [11].

In such a model it might be supposed that the state of our consciousness is represented by the mental wave function ψ_C. By using Freud's terminology we can say that one has classical *sub-consiousness* and quantum-like consiousness, cf. [10], [11]. QL-consiousness is represented by the mental wave function ψ_C. The crucial point is that in this model consciousness is created through neglecting an essential volume of information contained in subconsciousness. Of course, this is not just a random loss of information. Information is selected through the algorithm presented in section 4: context C is projected onto ψ_C.

The (classical) mental state of subconsiousness evolves with time $C \rightarrow C(t)$. This dynamics induces dynamics of the mental wave function $\psi(t) = \psi_{C(t)}$ in the complex Hilbert space, see [9] for the mathematical details.

Postulate QLR. *The brain is able to create the QL-representation of mental contexts, $C \rightarrow \psi_C$*

(by using the algorithm based on the formula of total probability with interference, see section 4).

8 Brain as quantum-like computer

We can speculate that the ability of the brain to create the QL-representation of mental contexts, see Postulate QLR, induces functioning of the brain as a quantum-like computer.

Postulate QLC. *The brain performs computation-thinking by using algorithms of quantum computing in the complex Hilbert space of mental QL-states.*

We emphasize that in our approach the brain is not quantum computer, but QL-computer. On one hand, QL-computer works totally in accordance with mathematical theory of quantum computations (so by using quantum algorithms). On the other hand, it is not based on superposition of individual mental states. The complex amplitude ψ_C representing a mental context C is a special probabilistic representation of information states of the huge neuronal ensemble. In particular, the brain is *macroscopic* QL-computer. Thus the QL-parallelism (in the opposite to conventional quantum parallelism) has a natural realistic base. This is real parallelism in working of millions of neurons. The crucial point is the way in which this classical parallelism is projected onto dynamics of QL-states. The QL-brain is able to solve NP-problems. But there is nothing mysterious in this ability: exponentially increasing number of operations is performed through involving of exponentially increasing number of neurons.

We pay attention that by coupling QL-parallelism to working of neurons we started to present a particular ontic model for QL-computations. We shall discuss it in more detail. Observables a and b are self-observations of brain. They can be represented as functions of the internal state of brain ω. Here ω is a parameter of huge dimension describing states of all neurons in brain: $\omega = (\omega_1, \omega_2, ..., \omega_N) : a = a(\omega), b = b(\omega)$. The brain is not interested in concrete values of the reference observables at fixed instances of time. The brain finds the contextual probability distributions $p_C^b(x)$ and $p_C^a(y)$ and creates the mental QL-state $\psi_C(x)$, see algorithm in section 4. Then it works with $\psi_C(x)$ by using algorithms of quantum computing. The crucial problem is to find mechanism of calculating of contextual probabilities. We think that they are frequency probabilities which are created in the brain in the following way.

There are two scales of time: a) internal scale; b) QL-scale. The internal scale is finer than the QL-scale. Each instant of QL-time t corresponds to an interval Δ of internal time τ. We might identify the QL-time with mental (psychological) time and the internal time with physical time. During the interval Δ of internal time the brain collects statistical data for self-observations of a and b.

Thus the internal state ω of the brain evolves as $\omega = \omega(\tau, \omega_0)$. At each instance of internal time τ there are performed nondisturbative self-measurements of a and b. These are realistic measurements: the brain gets values $a(\omega(\tau, \omega_0)), b(\omega(\tau, \omega_0))$. By finding frequencies of realization of fixed values for $a(\omega(\tau, \omega_0))$ and $b(\omega(\tau, \omega_0))$ the brain obtains the frequency probabilities $p_C^a(x)$ and $p_C^b(y)$. These probabilities are related to the instant of QL-time time t corresponding to the interval of internal time $\Delta : p_C^b(t, x)$ and $p_C^a(t, y)$.

For example, a and b can be measurements over different domains of brain. It is supposed that the brain can "feel" probabilities (frequencies) $p_C^b(x)$ and $p_C^a(y)$, but not able to "feel" the simultaneous probability distribution $p_C(x, y) = P(b = x, a = y/C)$. This is not the problem of mathematical

existence of such a distribution.[3] This is the problem of integration of statistics of observations from different domains of the brain. By using the QL-representation based only on probabilities $p_C^b(x)$ and $p_C^a(y)$ the brain could be able to escape integration of information about *individual self-observations* of variables a and b related to spatially separated domains of brain. The brain need not couple these domains at each instant of internal time τ. It couples them only once in the interval Δ through the contextual probabilities $p_C^b(x)$ and $p_C^a(y)$. This induces the huge saving of time.

9   Quantum-like information dynamics

The mental wave function $\psi(t)$ evolves in the complex Hilbert space (space of probability amplitudes, see section 4). The straightforward generalization of quantum mechanics would imply the *linear* Schrödinger equation:

$$i\frac{d\psi(t)}{dt} = \hat{\mathcal{H}}\psi(t), \psi(0) = \psi_0, \tag{6}$$

where $\hat{\mathcal{H}} : H \to H$ is a self-adjoint operator in the Hilbert space H of mental QL-states. However, the Växjö model predicts [9] broader spectrum of evolutions in the Hilbert space (induced by evolutions of contexts). We could not go deeply into mathematical details and only remark that in general the contextual dynamics $C \to C(t)$ can induce *nonlinear* evolutions in H :

$$i\frac{d\psi(t)}{dt} = \hat{\mathcal{H}}(\psi(t)), \psi(0) = \psi_0, \tag{7}$$

where $\hat{\mathcal{H}} : H \to H$ is a nonlinear map. It is important to point out that even the nonlinear dynamics in the Hilbert state space induced by a contextual dynamics is *unitary:* $(\psi(t), \psi(t)) = (\psi(0), \psi(0))$.

In principle,*there are no a priory reasons to assume that the mental quantum-like dynamics should always be linear!* It might be that nonlinearity of the Hilbert space dynamics is the distinguishing feature of cognitive systems. However, at the present time this is just a speculation.

10   Classical and quantum projections of information prespace

We constructed projections of special classes of contexts, C^{tr} and C^{hyp}, to complex and hyperbolic Hilbert spaces. We pay attention that in general $C^{\mathrm{tr}} \cup C^{\mathrm{hyp}} \neq F$. There exist contexts which could not be projected to the complex nor hyperbolic Hilbert space. Quantum mechanics is not complete; moreover, even both quantum models (complex and hyperbolic) do not give the complete image of prespace reality. How could we complete our picture of prespace? We should be able to find a new fundamental physical variable u and take also its conjugate variable v such that they induce realistic observables $J(u)$ and $J(v)$. It is supposed that these observables are nonreducible to the position and momentum observables. Since in general $C_{b/a}^{\mathrm{tr}} \neq C_{u/v}^{\mathrm{tr}}$ and $C_{b/a}^{\mathrm{hyp}} \neq C_{u/v}^{\mathrm{hyp}}$, we get in the complex and hyperbolic Hilbert spaces images of new prespace contexts (by using the maps

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[3]We recall that, since we consider only two realistic observables, there is no direct contradiction with Bell's inequality.

$J^{u/v} : C^{\text{tr}}_{u/v} \to H$ and $J^{u/v} : C^{\text{hyp}}_{u/v} \to H^{\text{hyp}}$). However, it might be that human beings could not even in principle observe physical variables nonreducible to position.

We also make remark about the von Neumann "No-GO" theorem [2]. Our model does not contradict to von Neumann's coclusion that there are no *dispersion free quantum states.* Of course, the Kolmogorov model on prespace contains dispersion free contexts for the reference observables (e.g. the position and momentum). In particular, the Heisenberg uncertainty relations are violated for them. But such contexts do not belong to the class of trigonometric contexts C^{tr}. Therefore they do not have images in the complex Hilbert space of quantum states. For example, a single point context $C_\omega = \{\omega\}$ is dispersion free, but does not belong to C^{tr}.

In our approach classicality is a joint feature of a context and reference observables. It is meaningless to speak about classical observables without relation to a context. A prespace context C can be called classical with respect to an observational model with reference observables a and b if in this model it is possible to find the joint probability distribution $p_C(x,y) = \mathbf{P}(b = x, a = y)$. Denote the set of classical contexts by the symbol C^{class}. We pay attention that there is a crucial difference between definitions of trigonometric and hyperbolic contexs and classical contexts. First two classes are defined in internally prespace terms (in the ontic model); the third class cannot be defined in the ontic model. Another important point is that we speak about the possibility "to find" and not (as many authors) about the purely mathematical existence of the joint probability distribution. In our model it always exists, but in general the observational model does not give us the possibility to find it.

Let us introduce the classical phase space corresponding to the reference observables: $Z = X \times Y$ (e.g. position–momentum space). Denote the set of probabilistic measures on the phase space by $M(Z)$. It is natural to represent classical contexts by elements of $M(Z) : J^{b,a}_{\text{class}} : C^{\text{class}} \to M(Z)$, $C \to p_C$. This is our interpretation of the classical statistical mechanics (considered as an epistemological model) as the image of the set of classical contexts in prespace . We emphasize that even classical projection induces the huge loss of information about contexts. In particular, each point $z = (x,y) \in Z$ which can be represented by the Dirac δ-measure is in fact the image of the huge domain in the prespace, namely $W_z = \{\omega \in \Omega : b(\omega) = x, a(\omega) = y\}$. Stationarity in the classical phase space is the image of very complex motions in the prespace (the same is valid for stationarity in the quantum state space).

The same model can be used for the mental reality. As was already remarked, here we have the problem of the choice of fundamental mental variables – analogues of position and momentum. It might be that there is no such fixed for ever observables and the brain can easily change representation of mental reality.

Finally, we make a short remark about time. Time is classical both in quantum and classical epistemological models in the following sense. It is assumed that beside the fundamental variable b and its conjugate a there is well defined a time variable $T : \Omega \to \mathbf{R}$. It is also possible to represent it by the realistic time-observable. Let context $C \in C^{\text{class}}$. Then it is assumed that there is well defined the joint probability distribution $p_C(t; x, y) = \mathbf{P}(T = t, b = x, a = y)$. It evolves according to the Liouville equation. Let now context $C \in C^{\text{tr}}$. It is assumed that for any reference observable there is defined a joint probability distribution with the time-observable: $p_C(t; x) = \mathbf{P}(T = t, b = x)$ and $p_C(t; y) = \mathbf{P}(T = t, a = y)$. These probability distributions determine (under some assumptions) the complex probability amplitude $\psi_C(t, x)$ which evolves (under some assumptions) according to the Schrödinger equation. In prespace there is no time! Time is created through a special realistic observable on the prespace. Finally, we remark that each instant of time t corresponds to a huge domain in the prespace, namely $W_t = \{\omega \in \Omega : T(\omega) = t\}$.

Thus time (as well as spatial coordinateis) is produced via projection of information prespace.

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