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Random spheres as a 1+1 dimensional field theory.

Rémi Léandre

Institut de Mathématiques. Université de Bourgogne. 21000. Dijon France.
E-mail: Remi.leandre@u-bourgogne.fr

Abstract: We construct non-gaussian Brownian spheres on R^d as a 1+1 dimensional field theory.

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1 Introduction

In string theory or in conformal field theory (See survey of Witten [51] or Gawedzki [18], [19],[20]), people look at a Riemannian surface Σ (the world-sheet) and at a Riemannian manifold M (the target space) and consider the set of maps $x(.)$ from Σ into M endowed with the formal probability measure:

$$d\mu(x(.)) = Z^{-1}\exp[-I(x(.))]dD(x(.)) \tag{1}$$

where $I(x(.))$ is the energy of the field $x(.)$, $dD(x(.))$ the formal Lebesgue measure on the set of fields and Z the partition function destinated to get a probability measure.
If we consider the case where $M = R$ with the flat Riemannian structure, we get a Gaussian measure, called the free field measure. Let G be the green kernel associated to the Laplace-Beltrami operator on Σ. Let S be the generic element of Σ. We have, in this case:

$$E[x(S)x(S')] = G(S, S') \tag{2}$$

and $G(S,S) = \infty$ in two dimension. This explains that we consider only smeared fields, and that the Gaussian measure of the free field lives on distributions in fact ([46]).
There is another way to construct non-linear random fields, parametrized by surfaces, which are continuous. It is the purpose of infinite dimensional processes on infinite dimensional manifolds, which comes from the work of Kuo ([23]), Beloposkaya-Daletskii ([8]) and Daletskii ([13]).
The simplest geometry for the world sheet is the case where we consider the cylinder: this corresponds to diffusion processes on the loop space of the manifold.
This can be seen in two ways:
-)Either we consider Ornstein-Uhlenbeck process on the loop space, by using Dirichlet forms theory ([6], [15], [27], [41]).
-)Or we consider Brownian motion on the loop space, in the sense of Airault-Malliavin ([3],[10], [11], [36]).
In these two cases, we say that we are in presence of a $1+1$ dimensional field theory, 1 for the dynamical time and 1 for the dimension of the internal time of the theory.
Some refinment of $1+1$ dimensional field theories were done in [12], [37] and [38], where we consider pants. It is a finite dimensional constrain, and traditional tools of stochastic analysis as quasi-sure analysis ([2], [48]) can be applied.
Until now, in order to understand a more complicated world-sheet, we were obliged to consider a $1+2$ dimensional field theory, that is processes on the set of maps from Σ into M ([28], [30], [31], [32], [33], [34], [39]) and the associated heat-kernel measure.
Our goal is to understand a more complicated world-sheet, that is the sphere (We start from the constant loop and we arrive at the constant loop) as a $1+1$ dimensional stochastic field theory. So we construct a process on the loop space of R^d (endowed with a non-trivial Riemannian structure, but **without cut-locus!**) starting from the constant loop and submitted to coming-up at the constant loop at time 1. It is an infinite dimensional constrain, and we cannot apply tools of quasi-sure analysis in order to do this work ([2], [48]).
We refer to the survey of Albeverio ([4]) for the relation between stochastic analysis and mathematical physics as well as the 3 surveys of Léandre about that ([27], [29], [40]).

2 The model

Let X_i be $m+1$ vector fields on R^d bounded with bounded derivatives of all orders. We suppose that the quadratic form $g(x)^{-1}$

$$\xi \to \sum_{i=1}^{m} < X_i(x), \xi >^2 \tag{3}$$

is uniformly invertible on R^d. Let $w^i, i = 1,..,m$ a R^m-valued Brownian motion. We can consider the Stratonovich differential equation issued from x:

$$dx_t(x) = \sum_{i=1}^{m} X_i(x_t(x))dw_t^i + X_0(x_t(x))dt \tag{4}$$

The law of $x_t(x)$ has a smooth density $q_t(x, y)$ with respect to the Lebesgue measure on R^d, strictly positive, and which is smooth in $t > 0, x, y$. If we constrain the diffusion $x_t(x)$ to be x at time 1, it is the solution of the Stratonovitch differential equation ([9], [45]):

$$dx_t(x,x) = \sum_{i=1}^{m} X_i(x_t(x,x))d\tilde{w}_t^i + X_0(x_t(x,x))dt \tag{5}$$

where

$$d\tilde{w}_t^i = dw_t^i + < X_i(x_t(x,x)), grad \log q_{1-t}(x_t(x,x),x) > dt \tag{6}$$

We consider the Hilbert space H of free loops h in R^m endowed with the Hilbert structure:

$$\int_{S^1} |h(s)|^2 ds + \int_{S^1} |d/dsh(s)|^2 ds = \|h\|^2 \tag{7}$$

Let ψ_s^i be the map $h \to h^i(s)$ where h^i is the coordinate of order i of h. It is a continuous linear form. Therefore there exists an element e_s^i of H such that

$$h(s) = < h, e_s^i > \tag{8}$$

Let us consider the Brownian motion $B_t(.)$ with values in H. $t \to B_t(s)$ are a family of R^m-valued Brownian motion indexed by $s \in S^1$. Moreover,

$$E[B_t^i(s)B_t^j(s')] = t < e_s^i, e_{s'}^j > \tag{9}$$

We deduce by Kolmogorov lemma ([44]), that the random field $(t, s) \to B_t(s)$ is almost surely Hoelder. We consider the solution $x_t(s, x)$ of (5) where we replace dw_t by $dB_t(s)$ and the associated bridge $x_t(s, x, x)$.
We suppose in the sequel that the following hypothesis is satisfied:
Hypothesis H: For all multi-index (α) of length equal to 2, we have for $t \leq 1$:

$$t(|\frac{\partial^{(\alpha)}}{\partial x^{(\alpha)}} \log q_t(x,y)| + |\frac{\partial^{(\alpha)}}{\partial y^{(\alpha)}} \log q_t(x,y)|) < C < \infty \tag{10}$$

We refer to [22] and to [42] about this hypothesis, which says more and less that the Riemannian manifold R^d endowed with the Riemannian metric g has no cut-locus.
Moreover, we can suppose that the diffusion $x_t(s, x)$ is the Brownian motion on R^d endowed with the Riemannian stucture $g(x)$, by choosing a suitable drift X_0.

3 The main theorem

Theorem III.1: the multiparameter process $(t,s) \to x_t(s,x,x)$ has a version which is Hoelder in s and t under **Hypothesis H**.
Lemma III.2: When $t \to 0$

$$P\{\sup_s |x_t(s,x,x) - x| > C\} \leq K' \exp[-K/t] \tag{11}$$

for $K > O$ depending only from $C > 0$
Proof: We see that if we put

$$U_t(s,s') = x_t(s,x,x) - x_t(s',x,x) \tag{12}$$

that

$$U_t(s,s') = \int_0^t O_u^1 U_u(s,s')du + \int_0^t O_u^2 U_u(s,s')\delta B_u(s) + \int_0^t O_u^3 \delta(B_u(s) - B_u(s')) \tag{13}$$

where O_u^2 and O_u^3 are bounded continuous adapted processes and $|O_u^1| \leq \frac{C}{1-u}$ by **Hypothesis H**. δ denotes the Itô differential. We deduce by Hoelder inequality and Burkholder-Davies-Gundy ineqality that for some k depending of p only:

$$E|U_t(s,s')|^p] \leq C|\log(1-t)|^k \int_0^t \frac{du}{1-u}|U_u(s,s')|^p + C|s-s'|^{p/2} \tag{14}$$

because

$$E[|B_t(s) - B_t(s')|^2] = tO(1)|s-s'| \tag{15}$$

when $s \to s'$
Let us recall that for arbitrary small ϵ

$$\int_0^t \frac{du}{(1-u)|\log(1-u)|^{k'}} \leq C(1-u)^{-\epsilon} \tag{16}$$

We deduce by Gronwall lemma that:

$$E[|U_t(s,s')|^p] \leq C|s-s'|^{p/2} \exp[K(1-t)^{-\epsilon}] \tag{17}$$

We deduce by Kolmogorov lemma ([44]) that $s \to x_t(s,x,x)$ is $1/2 - \epsilon'$ Hoelder with Hoelder norm in L^p bounded by $\exp[K(1-t)^{-\epsilon}]$.
We consider $\exp[C_1(1-t)^{-1}]$ sites s_i on the circle. It is classical that $P\{|x_t(s_i,x,x) - x| > C]$ is bounded by $\exp[-D(1-t)^{-1}]$ when $t \to 0$. We choose C_1 small with respect of D. We get:
(18)

$$P\{\sup_s |x_t(s,x,x) - x| > C\} \leq \sum_i P\{|x_t(s_i,x,x) - x| > C\} + P\{\|x_t(.,x,x)\|_{1/2-\epsilon'} > \exp[E(1-t)^{-1}]\}$$

for some convenient $E > 0$ and where $\|.\|_{1/2-\epsilon'}$ denotes the Hoelder norm on the circle with exponent $1/2 - \epsilon'$.

We deduce the estimate
(19)
$$P\{\sup_s |x_t(s,x,x)-x| > C\} \leq \exp[C_1(1-t)^{-1}]\exp[-D(1-t)^{-1}]+\exp[-E(1-t)^{-1}]\exp[K(1-t)^{-\epsilon}]$$

The result because ϵ is small and because C_1 is strictly smaller than D.
$\diamondsuit$
We consider a smoth function h from R into R^+ equal to 0 in a neighborhood of 0 and equal to 1 in a neighborhood of ∞. We denote

$$z_t = h(\sup_s |x_t(s,x,x)-x|) \tag{20}$$

From Lemma III.2, we deduce:
Lemma III.3: $Z = \int_0^1 \frac{z_t}{(1-t)}dt$ is almost surely finite.
Let us denote:

$$\sup_s |x_t(s,x,x)-x| \wedge C = z'_t \tag{21}$$

By a small modification of Lemma III. 2, we have if $\alpha < 1/2$

$$P\{z'_t > (1-t)^\alpha\} \leq \exp[-C(1-t)^{-\epsilon"}] \tag{22}$$

such that $Z' = \int_0^1 \frac{z'_t}{(1-t)}dt$ is almost surely finite.
Lemma III.4: If $Z + Z' < n$, there exist a constant $C(n,p)$ still denoted by C such that:

$$E[|x_t(s,x,x) - x_{t'}(s',x,x)|^p]^{1/p} \leq C(|s-s'|^{1/2} + |t-t'|^{1/2}) \tag{23}$$

Proof: We look at (13). We write

$$O^1_u = O^1_u z_u + O^1_u(1-z_u) = V^1_u + V^2_u \tag{24}$$

We have $|V^1_u| \leq C(1-u)^{-1}z_u$. We work in in normal system of coordinates of x for the Riemannian structure such that when $t \to 0$ we have the asymptotic expansion ([21], [26], [25], [50])

$$q_t(y,x) = t^{-d/2}\exp[-d^2(x,y)/2t]\sum c_i(y,x)t^i \tag{25}$$

if $|x-y| < C$. This asymptotic expansion works for the derivatives of $q_t(x,y)$ too, by taking derivatives in x and y of the asymptotics expansion (This implies that $c_i(x,y)$ is smooth in x and y). We write in normal coordinates $d^2(x,y) \sim |x-y|^2$ where $d^2(x,y)$ is the Riemannian metric on R^d for the Riemannian structure g. We deduce that

$$V^2_u(U_t(s,s')) = -\frac{1}{1-t}U_t(s,s') + \frac{A_t}{1-t}U_t(s,s') \tag{26}$$

where

$$|A_t| \leq C(\sup_s |x_t(s,x,x)_x| \wedge C) = Cz'_t \tag{27}$$

We deduce that $U_t(s,s')$ satisfies

(28)

$$\delta U_t(s,s') = -\frac{1}{1-t}U_t(s,s')dt + \frac{A_t}{1-t}U_t(s,s')dt + V_t^1 U_t(s,s')dt + O_t^2 U_t(s,s')\delta B_t(s) + O_t^3\delta(B_t(s)-B_t(s'))$$

We put $U_t(s,s') = (1-t)C_t(s,s')$. $C_0(s,s') = 0$ such that $C_t(s,s')$ satisfies the differential equation:

$$(29)\quad \delta C_t(s,s') = \frac{A_t}{(1-t)}C_t(s,s')dt + V_t^1 C_t(s,s')dt + O_t^2 C_t(s,s')\delta B_t(s) + \frac{O_t^3}{(1-t)}\delta(B_t(s)-B_t(s'))$$

We deduce by Gronwall inequality since $Z+Z' \leq n$ that

$$(30)\qquad |C_t(s,s')| \leq C'_n \sup_{v\leq t}(|\int_0^v O_u^2 C_u(s,s')\delta B_t(s) + \frac{O_u^3}{1-u}\delta(B_u(s)-B_u(s'))|)$$

By Gronwall lemma and Burkholder -Davis-Gundy inequality, we deduce that

$$(31)\qquad \|C_t(s,s')\|_{L^p} \leq C"_n|s-s'|^{1/2}(1-t)^{-1}$$

Therefore

$$(32)\qquad \|U_t(s,s')\|_{L^p} \leq C"_n|s-s'|^{1/2}$$

Moreover, we have clearly the inequality:

$$(33)\qquad \|x_t(s,x,x) - x_{t'}(s,x,x)\|_{L^p} \leq C|t-t'|^{1/2}$$

$\diamondsuit$

Proof of Theorem III.1: If $Z+Z' < n$, we have an Hoelder version of the random field $(t,s) \to x_t(s,x,x)$. Since $Z+Z'$ is almost surely finite the result arises.

$\diamondsuit$

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