Liouville Dynamics, Distinguishability of States, and the Conservation of Classical Information.

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Abstract: We advance a general formalism for the discussion of classical analogues of quantum mechanical, information-related impossible operations, such as the celebrated no-cloning theorem. Our approach is based on the conservation of information distances between pairs of time-dependent solutions of Liouville equation. On the basis of Fisher's information measure, we show that the aforementioned classically forbidden operations admit of an interpretation in terms of statistical estimation theory.

Keywords: Physics of Information; Liouville Equation; No-cloning Theorem.

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1 Introduction

The physics of information has been the focus of intense research efforts in recent years [1-12]. Information-related concepts have been successfully applied to the study of diverse physical problems ranging from quantum mechanical aspects of the second law of thermodynamics [9] to the evolution of inhomogeneous cosmological models [10]. Interest in the Physics of Information has been greatly stimulated by the discovery of novel and counter-intuitive ways of processing and transmitting information allowed by the laws of quantum mechanics [11, 12]

Some basic features of quantum physics that are relevant for the processing of information admit of classical counterparts. Classical analogues of entanglement [13] and quantum search algorithms [14] as well as classical dynamical settings leading to non-Boolean logics [15] have been discovered. It has also recently been shown [16, 17] that the Liouville dynamics describing the evolution of classical statistical ensembles exhibits a classical counterpart of the celebrated quantum non-cloning theorem [18, 19]. Besides quantum cloning, however, there are other important examples of information-related processes forbidden by the laws of quantum mechanics, e.g., quantum deleting [20] and quantum disentangling [21]. The physical impossibility of those quantum operations has profound implications for both quantum information theory and quantum physics in general, being nowadays the focus of intensive research [22-25].

The aim of this Contribution is to propose a general formalism for the analysis of classical counterparts of some of the aforementioned quantum impossible processes. We focus on classical analogues of the non-cloning and the no-deleting theorems, which we deduce from the Liouville dynamics that governs the evolution of statistical ensembles. We will prove that the corresponding classically forbidden operations are not compatible with an important property of Liouville dynamics: the conservation of the (Kullback-Leibler [26] and related) information distances. The impossibility of quantum operations such as universal quantum cloning is, no doubt, one of the most fundamental properties of quantum information. However, not all aspects of the concomitant impossibility theorems are inherently quantum mechanical. Investigating classical analogues of the alluded to theorems can help to identify their essentially quantum mechanical features, as opposed to those aspects that may arise within purely classical probabilistic settings. In order to make progress in this direction, it is useful to study specific probabilistic classical scenarios admitting a no cloning, or a no deleting theorems. Here we consider such an scenario, based upon the Liouville dynamics governing the evolution of statistical ensembles of classical dynamical systems. This line of inquiry may contribute to a deeper understanding of the relationship between classical and quantum mechanical probabilities. On the other hand, the investigation of basic properties of the solutions of Liouville equation is of general interest, due to the mathematical and physical importance of this equation.

2 Liouville Equation and Conserved Entropic Distances

We are going to consider general classical deterministic dynamical systems governed by equations of motion of the form

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}(\boldsymbol{x}), \qquad \text{with} \quad \boldsymbol{x}, \boldsymbol{v} \in R^N, \tag{1}$$

where $\boldsymbol{x}$ denotes a point in the concomitant N-dimensional phase space. Hamiltonian dynamics constitutes a particular instance of (1). In the case of a Hamiltonian system with n degrees of freedom we have $N = 2n$, $\boldsymbol{x} = (q_1, \ldots, q_n, p_1, \ldots, p_n)$, $v_i = \partial H/\partial p_i \;\; (i = 1, \ldots, n)$, and $v_{i+n} = -\partial H/\partial q_i \;\; (i = 1, \ldots, n)$, where the q_i and the p_i stand for generalized coordinates and momenta, respectively. Note that Hamiltonian dynamics exhibits the important feature of being divergence-free

$$\nabla \cdot \boldsymbol{v} = 0 \tag{2}$$

.
Our present considerations are neither restricted to Hamiltonian systems nor to systems with divergence-free flows.

The dynamics of a statistical ensemble of systems evolving according to equation (1) is described by a time-dependent probability distribution $\mathcal{P}(\boldsymbol{x}, t)$ whose evolution is given by the Liouville equation,

$$\frac{\partial}{\partial t}\mathcal{P} \,+\, \nabla \cdot (\boldsymbol{v}\mathcal{P}) \,=\, 0. \tag{3}$$

The study of time dependent, information-related aspects of Liouville equation, is usually focused on the time evolution of the entropy,

$$S[\mathcal{P}] \;=\; - \int \mathcal{P}\, \ln \mathcal{P}\, d\boldsymbol{x}. \tag{4}$$

The time derivative of S is given by the mean value of the flow's divergence [27],

$$\begin{aligned} \frac{dS}{dt} &= \int (\nabla \cdot \mathbf{v})\, \mathcal{P}\, d\boldsymbol{x} \\ &= \langle \nabla \cdot \mathbf{v} \rangle. \end{aligned} \tag{5}$$

Consequently, S is constant only for divergenceless dynamical systems. However, deterministic dynamical systems with $\nabla \cdot \mathbf{v} \neq 0$ exhibit a time dependent entropy.

Let us now consider a functional depending on two time dependent solutions of Liouville equation, $\mathcal{P}_1$ and $\mathcal{P}_2$,

$$G[\mathcal{P}_1, \mathcal{P}_2] \;=\; \int d\boldsymbol{x}\, \mathcal{P}_1\, g\left[\frac{\mathcal{P}_1}{\mathcal{P}_2}\right], \tag{6}$$

where $g\,[\cdots]$ denotes an arbitrary function (we assume that the integral in (6 converges). In turns out that, for general deterministic dynamical systems (1), the functional (6) is preserved by the Liouville dynamics [17],

$$\frac{dG}{dt} \;=\; 0. \tag{7}$$

Hence, depending on the explicit choice of the function $g\,[\cdots]$, the functional $G\,[\mathcal{P}_1, \mathcal{P}_2]$ provides a convenient way to measure time-invariant relations (distances) between $\mathcal{P}_1$ and $\mathcal{P}_2$. We stress the fact that the conservation of G, given by equation (7), holds true for any deterministic dynamical system, and not only for divergenceless systems (as happens with the entropy S). In this sense,

the behaviour of G constitutes a clear difference between deterministic and stochastic dynamics. If we add, as in the Fokker-Planck equation, a diffusion term (originated in stochastic contributions to the dynamics) to the Liouville equation, functionals like G are no longer conserved [28].
In order to illustrate the functional (6), notice that the special case

$$g\left[\frac{\mathcal{P}_1}{\mathcal{P}_2}\right]=\ln\frac{\mathcal{P}_1}{\mathcal{P}_2} \overset{G\to K}{\Longrightarrow} K(\mathcal{P}_1,\mathcal{P}_2)=\int d\boldsymbol{x}\,\mathcal{P}_1\ln\frac{\mathcal{P}_1}{\mathcal{P}_2} \tag{8}$$

gives the Kullback-Leibler distance and that

$$g\left[\frac{\mathcal{P}_1}{\mathcal{P}_2}\right]=\sqrt{\frac{\mathcal{P}_2}{\mathcal{P}_1}} \overset{G\to O}{\Longrightarrow} O(\mathcal{P}_1,\mathcal{P}_2)=\int d\boldsymbol{x}\,\sqrt{\mathcal{P}_1\mathcal{P}_2} \tag{9}$$

corresponds to the overlap between $\mathcal{P}_1$ and $\mathcal{P}_2$. This overlap is closely related to a frequently used statistical distance between probability distributions [29] given by $\cos^{-1}\{O(\mathcal{P}_1,\mathcal{P}_2)\}$.
In the present contribution we are going to consider the mono-parametric family of functionals introduced by Tsallis [30],

$$I_q\left[\mathcal{P}_1,\mathcal{P}_2\right] = \int d\boldsymbol{x}\,\mathcal{P}_1\,\frac{\left[\mathcal{P}_1/\mathcal{P}_2\right]^{q-1}-1}{q-1}, \tag{10}$$

parameterized by the real parameter $0<q\leq 1$. For this range of q-values we have

$$I_q\left[\mathcal{P}_1,\mathcal{P}_2\right] \geq 0. \tag{11}$$

Instead of working directly with I_q, it is going to prove convenient to use the functional

$$\begin{aligned} D_q\left[\mathcal{P}_1,\mathcal{P}_2\right] &= 1+(q-1)\,I_q\left[\mathcal{P}_1,\mathcal{P}_2\right] \\ &= \int d\boldsymbol{x}\,\mathcal{P}_1^q\,\mathcal{P}_2^{1-q}. \end{aligned} \tag{12}$$

It is plain from Hölder's inequality that

$$D_q\left[\mathcal{P}_1,\mathcal{P}_2\right] \leq 1, \tag{13}$$

with the equality sign holding only if $\mathcal{P}_1=\mathcal{P}_2$ (remember that both probability distributions $\mathcal{P}_1$ and $\mathcal{P}_2$ are normalized to 1). The functionals $I_q\left[\mathcal{P}_1,\mathcal{P}_2\right]$ and $D_q\left[\mathcal{P}_1,\mathcal{P}_2\right]$ are also connected with the Kullback distance and the overlap,

$$\lim_{q\to 1} I_q\left[\mathcal{P}_1,\mathcal{P}_2\right] = K(\mathcal{P}_1,\mathcal{P}_2)\,, \tag{14}$$

and

$$D_{1/2}\left[\mathcal{P}_1,\mathcal{P}_2\right] = O(\mathcal{P}_1,\mathcal{P}_2)\,. \tag{15}$$

As a matter of fact, the functional I_q was originally advanced as a non-extensive generalization of the standard Kullback-Leibler distance [30]. Classical analogs of the no-cloning and related theorems were discussed by us in [16, 17] on the basis of the Kullback and the overlap measures. Here we are going to show that both approaches can be unified an generalized in terms of the measures I_q and D_q.

Suppose we have a composite system consisting of two statistically independent subsystems a and b described by a factorized joint probability distribution $\mathcal{P} = \mathcal{P}^{(a)}\mathcal{P}^{(b)}$. It follows from (10) that the I_q distance between two such distributions verifies

$$I_q\left[\mathcal{P}_1, \mathcal{P}_2\right] = I_q\left[\mathcal{P}_1^{(a)}, \mathcal{P}_2^{(a)}\right] + I_q\left[\mathcal{P}_1^{(b)}, \mathcal{P}_2^{(b)}\right] + (q-1)\, I_q\left[\mathcal{P}_1^{(a)}, \mathcal{P}_2^{(a)}\right] I_q\left[\mathcal{P}_1^{(b)}, \mathcal{P}_2^{(b)}\right], \tag{16}$$

while the D_q functional complies with

$$D_q\left[\mathcal{P}_1, \mathcal{P}_2\right] = D_q\left[\mathcal{P}_1^{(a)}, \mathcal{P}_2^{(a)}\right] D_q\left[\mathcal{P}_1^{(b)}, \mathcal{P}_2^{(b)}\right]. \tag{17}$$

That is, for factorized probability distributions the total distance D_q is equal to the product of the individual D_q distances between the two subsystems. In the limit case $q \to 1$, equation (16) reduces to the additivity property of the Kullback distance,

$$K(\mathcal{P}_1, \mathcal{P}_2) = K\left(\mathcal{P}_1^{(a)}, \mathcal{P}_2^{(a)}\right) + K\left(\mathcal{P}_1^{(b)}, \mathcal{P}_2^{(b)}\right). \tag{18}$$

On the other hand, the factorizability of the overlap for factorized probability distributions,

$$O(\mathcal{P}_1, \mathcal{P}_2) = O\left(\mathcal{P}_1^{(a)}, \mathcal{P}_2^{(a)}\right) O\left(\mathcal{P}_1^{(b)}, \mathcal{P}_2^{(b)}\right), \tag{19}$$

constitutes a particular instance of (17) for $q = 1/2$.

3 Classical No-Cloning and No-Deleting Theorems

We are now going to apply the distance (12) to formulate a classical analogue of the quantum no-cloning theorem. The basis of our argument is to study the D_q distance between two solutions $j = 1, 2$ of the Liouville equation of a tri-partite system composed of a machine (m), a source system (s), and a target system (t). The concomitant initial states (probability distributions) read

$$\mathcal{P}_j = \mathcal{P}_{\text{start}}^{(m)}\left(\boldsymbol{x}^{(m)}\right) \mathcal{P}_j^{(s)}\left(\boldsymbol{x}^{(s)}\right) \mathcal{P}_{\text{blank}}^{(t)}\left(\boldsymbol{x}^{(t)}\right). \tag{20}$$

The corresponding final distributions are denoted by $\mathcal{Q}_j$. Creating an exact copy of the source into the target implies that, for instance, the marginal distributions (of the final states) become

$$\int d\boldsymbol{x}^{(m)} \mathcal{Q}_j = \mathcal{P}_j^{(s)}\left(\boldsymbol{x}^{(s)}\right) \mathcal{P}_j^{(s)}\left(\boldsymbol{x}^{(t)}\right). \tag{21}$$

Due to (17) and assuming that $\mathcal{P}_{\text{start}}^{(m)}$ and $\mathcal{P}_{\text{blank}}^{(t)}$ are normalized, the D_q distance between the initial states (20) results in

$$D_q\left(\mathcal{P}_1, \mathcal{P}_2\right) = D_q\left(\mathcal{P}_1^{(s)}, \mathcal{P}_2^{(s)}\right) \tag{22}$$

On the other hand, using the inequality of Hölder, we find that the distance between the final states verify,

$$D_q[\mathcal{Q}_1, \mathcal{Q}_2] \quad = \quad \int d\boldsymbol{x}^{(s)} d\boldsymbol{x}^{(t)} d\boldsymbol{x}^{(m)} \mathcal{Q}_1^q \mathcal{Q}_2^{1-q}$$

$$
\begin{aligned}
&\leq \int d\boldsymbol{x}^{(s)} d\boldsymbol{x}^{(t)} \left[\int d\boldsymbol{x}^{(m)} \mathcal{Q}_1 \right]^q \left[\int d\boldsymbol{x}^{(m)} \mathcal{Q}_2 \right]^{1-q} \\
&\overset{(21)}{=} \int d\boldsymbol{x}^{(s)} \left[\mathcal{P}_1^{(s)} \left(\boldsymbol{x}^{(s)} \right) \right]^q \left[\mathcal{P}_2^{(s)} \left(\boldsymbol{x}^{(s)} \right) \right]^{1-q} \\
&\quad \times \int d\boldsymbol{x}^{(t)} \left[\mathcal{P}_1^{(s)} \left(\boldsymbol{x}^{(t)} \right) \right]^q \left[\mathcal{P}_2^{(s)} \left(\boldsymbol{x}^{(t)} \right) \right]^{1-q} \\
&\overset{(22)}{=} \left(D_q[\mathcal{P}_1, \mathcal{P}_2] \right)^2 \leq D_q[\mathcal{P}_1, \mathcal{P}_2]
\end{aligned} \tag{23}
$$

If $\mathcal{P}_1^{(s)} \neq \mathcal{P}_2^{(s)}$ and $D_q\left(\mathcal{P}_1^{(s)}, \mathcal{P}_2^{(s)}\right) > 0$, the inequalities in (23) are strict inequalities, and the cloning process is incompatible with the conservation of the D_q distance under Liouville dynamics. Consequently, the conservation of D_q implies that it is not possible to implement a universal cloning process on the basis of Liouville dynamics. The above argument does not hold when $D_q\left(\mathcal{P}_1^{(s)}, \mathcal{P}_2^{(s)}\right) = 0$. In other words, the cloning of states with vanishing D_q distance (that is, non-overlapping states) is not excluded.

A similar line of reasoning can be followed using the Kullback distance [17]. It follows from (18) and (21) that the distance between the final states $\mathcal{Q}_j$ complies with the inequality

$$
K(\mathcal{Q}_1, \mathcal{Q}_2) \geq 2K\left(\mathcal{P}_1^{(s)}, \mathcal{P}_2^{(s)}\right) = 2K(\mathcal{P}_1, \mathcal{P}_2)\,. \tag{24}
$$

If $K\left(\mathcal{P}_1^{(s)}, \mathcal{P}_2^{(s)}\right) \neq 0$, this inequality is clearly inconsistent with the conservation of the Kullback measure — see [16] for more details. The argument based on the D_q measure has the advantage, over the argument based on K, that D_q is still well defined when either of the distributions to be compared (or, even, both of them simultaneously) vanish in some regions of phase space. Summing up, (23) and (24) are both incompatible with the conservation of (12) and (8), respectively, provided that the dynamical evolution of the relevant probability distributions is given by the Liouville equation (3).

Some remarks are here in order concerning our particular choices for the forms of the initial and the final states of the cloning process. We have assumed initial factorized states (20). It is a natural assumption that initially the source system, whose state is to be copied into the target system, is statistically independent of (i.e., uncorrelated to) the target and the copy machine. In a universal copy machine, the initial states of the target and copy machine have to be independent of the initial state (to be copied) of the source system. On the other hand, it would be possible to assume an initial correlation between the initial states of the copy machine and the target system. But this correlated, joint state of the copy machine-target system has to be independent of the initial state of the source (and be the same for all possible initial states of the source). Assuming such an initially correlated state for the copy machine-target system does not affect our present arguments, since the initial distance D_q between two initial states of the complete, tri-partite system (i.e., copy machine-source-target system) is still given by equation (22). As for the final distribution of the tripartite system, we are not assuming that the associated (total) joint distribution is fully factorizable. The distribution corresponding to the copy machine may end up correlated with the distribution associated with the bipartite source-target system. However, a successful cloning process must be such that the marginal distribution associated with this bipartite system is factorizable (see eq. (21)). When cloning the original ensemble distribution associated with the source system, we want to end up with two statistically independent systems (i.e., source and target) described by the same probability distribution. The aim of the cloning process is to have access to this two copies, in order to be able to get more information about the original distribution than the one we can get from just one copy. What we have shown is that such a cloning process is not compatible with the Liouvillian dynamics associated with deterministic dynamical systems.

An alternative approach to the cloning process is to consider initial states of the source system belonging to a family of probability distributions functions $\mathcal{P}(\boldsymbol{x};\lambda)$, characterized by a single parameter λ [17]. The distance (6) between two close distributions belonging to this family is given by [28]

$$G[\mathcal{P}(\boldsymbol{x};\lambda)\,,\mathcal{P}(\boldsymbol{x};\lambda{+}\varepsilon)] = C\varepsilon^2 I\left[\mathcal{P}(\boldsymbol{x};\lambda)\right] + \mathcal{O}\left(\varepsilon^3\right), \tag{25}$$

where C is a constant depending on the form of the function $g(\cdots)$. The form $I[\cdots]$ is Fisher's information measure, that reads

$$I\left[\mathcal{P}\left(\boldsymbol{x};\lambda\right)\right] \;=\; \int \frac{1}{\mathcal{P}}\left(\frac{\partial\mathcal{P}}{\partial\lambda}\right)^2 d\boldsymbol{x}. \tag{26}$$

Fisher's information is a non-negative quantity that plays a key role in statistical estimation theory. Indeed, if one tries to infer the parameter λ from one sample $\boldsymbol{x}$ chosen from the distribution $\mathcal{P}$, then estimation theory asserts that the mean squared error E^2 for the (unbiased) estimation of λ obeys the Cramer-Rao bound [4]

$$E^2 \;\geq\; 1/I\left[\mathcal{P}\left(\boldsymbol{x};\lambda\right)\right], \tag{27}$$

in which equality is achieved for the "best" possible or *efficient* estimator. Coming back to the cloning problem, we assume that the initial states of the composite system involved in the process obey the form

$$\mathcal{P}_\lambda = \mathcal{P}^{(m)}_{\text{start}}\left(\boldsymbol{x}^{(m)}\right)\mathcal{P}^{(s)}\left(\boldsymbol{x}^{(s)};\lambda\right)\mathcal{P}^{(t)}_{\text{blank}}\left(\boldsymbol{x}^{(t)}\right). \tag{28}$$

A successful cloning operation should yield final states

$$\mathcal{Q}_\lambda = \mathcal{P}^{(m)}\left(\boldsymbol{x}^{(m)};\lambda\right)\mathcal{P}^{(s)}\left(\boldsymbol{x}^{(s)};\lambda\right)\mathcal{P}^{(s)}\left(\boldsymbol{x}^{(t)};\lambda\right). \tag{29}$$

Because Fisher's information is preserved under Liouville evolution, we have [17]

$$I\left[\mathcal{P}^{(s)}\left(\lambda\right)\right] \;=\; 2I\left[\mathcal{P}^{(s)}\left(\lambda\right)\right] + I\left[\mathcal{P}^{(m)}\left(\lambda\right)\right], \tag{30}$$

which is clearly impossible provided $I\left[\mathcal{P}^{(s)}(\lambda)\right]\neq 0$. Interestingly, this result can be re-interpreted in terms of statistical estimation theory: we compare the Fisher measures associated with the initial state (28) and the final state (29), respectively, and obtain

$$I\left[\mathcal{Q}_\lambda\right] \;\geq\; 2I\left[\mathcal{P}_\lambda\right]; \tag{31}$$

if the transformation from (28) to (29) were possible, then we would be able to use the final state $\mathcal{Q}_\lambda$ to estimate the parameter λ with an optimum mean squared error (assuming an efficient estimator)

$$E^2[\mathcal{Q}_\lambda] \;=\; \frac{1}{2I\left[\mathcal{P}_\lambda\right] + I\left[\mathcal{P}^{(m)}(\lambda)\right]} \;<\; \frac{1}{I\left[\mathcal{P}_\lambda\right]}. \tag{32}$$

This equation contradicts the Cramer-Rao inequality associated with the initial states $\mathcal{P}_\lambda$, because (27) provides a bound for the mean squared error associated with every (unbiased) estimation procedure. Note that the conservation of Fisher's information implies that the "distinguishability" of phase space ensembles does not change under Liouvillian evolution. On the contrary, final states generated by a universal cloning machine would be more "distinguishable" than the concomitant initial states.

The so-called non-deleting theorem also admits of a classical counterpart. In this case we assume that the initial states of both the source and the target systems are described by the same probability distribution. Hence, the corresponding initial joint distribution of the tri-partite system reads

$$\mathcal{P}_j \;=\; \mathcal{P}^{(m)}_{\text{start}}\left(\boldsymbol{x}^{(m)}\right)\mathcal{P}^{(s)}_j\left(\boldsymbol{x}^{(s)}\right)\mathcal{P}^{(s)}_j\left(\boldsymbol{x}^{(t)}\right). \tag{33}$$

The aim of the process is to delete information of the target system against that of the source system, bringing the probability distribution of the former to a blank state so that final joint distribution becomes

$$\mathcal{Q}_j \;=\; \mathcal{P}^{(m)}_j\left(\boldsymbol{x}^{(m)}\right)\mathcal{P}^{(s)}_j\left(\boldsymbol{x}^{(s)}\right)\mathcal{P}_{\text{blank}}\left(\boldsymbol{x}^{(t)}\right). \tag{34}$$

Assuming $D_q\left(\mathcal{P}^{(s)}_1,\mathcal{P}^{(s)}_2\right) \neq 0$, the conservation of the D_q measure yields

$$D_q\left(\mathcal{P}^{(s)}_1,\mathcal{P}^{(s)}_2\right) \;=\; D_q\left(\mathcal{P}^{(m)}_1,\mathcal{P}^{(m)}_2\right). \tag{35}$$

implying that the information deleted from the target system is entirely transferred into the final state of the deleting machine. Similarly, the conservation of the Kullback distance leads to

$$K\left(\mathcal{P}^{(s)}_1,\mathcal{P}^{(s)}_2\right) = K\left(\mathcal{P}^{(m)}_1,\mathcal{P}^{(m)}_2\right). \tag{36}$$

Again, all the information distance between the to be deleted states is transfered into the information distance between the final states of the deleting machine. In this sense, information distance can not be deleted. If the final state of the target system is given by a standard "blank" state, the information that was lost goes into the distribution associated with the copy machine. This is the essence of the classical (and also the quantum; see [20]) no-deleting theorem.

It is of interest to extend the present discussion to the case in which the initial state of the source-target system cannot be factorized. We consider an initial state

$$\begin{aligned}\mathcal{P}_j \;&=\; \mathcal{P}^{(m)}_{\text{start}}\left(\boldsymbol{x}^{(m)}\right)\mathcal{P}^{(s,t)}_j\left(\boldsymbol{x}^{(s)},\boldsymbol{x}^{(t)}\right)\\ &=\; \mathcal{P}^{(m)}_{\text{start}}\left(\boldsymbol{x}^{(m)}\right)\mathcal{P}^{(s)}_j\left(\boldsymbol{x}^{(s)}\right)h\left(\boldsymbol{x}^{(s)},\boldsymbol{x}^{(t)}\right),\end{aligned} \tag{37}$$

where $\mathcal{P}^{(s,t)}_j\left(\boldsymbol{x}^{(s)},\boldsymbol{x}^{(t)}\right)$ and $h\left(\boldsymbol{x}^{(s)},\boldsymbol{x}^{(t)}\right)$ are symmetrical functions of their respective arguments. $\mathcal{P}^{(s)}_j\left(\boldsymbol{x}^{(s)}\right) = \int \mathcal{P}^{(s,t)}_j\left(\boldsymbol{x}^{(s)},\boldsymbol{x}^{(t)}\right)d\boldsymbol{x}^{(t)}$ is the marginal probability distribution for $\boldsymbol{x}^{(s)}$ and $h\left(\boldsymbol{x}^{(s)},\boldsymbol{x}^{(t)}\right)$ denotes the conditional probability distribution of $\boldsymbol{x}^{(t)}$ for a given value of $\boldsymbol{x}^{(s)}$ (irrespective j). Given these specific assumptions, the transformation from the initial state (37) into a final state $\mathcal{P}^{(m)}_{\text{final}}\left(\boldsymbol{x}^{(m)}\right)\mathcal{P}^{(s)}_j\left(\boldsymbol{x}^{(s)}\right)\mathcal{P}_{\text{blank}}\left(\boldsymbol{x}^{(t)}\right)$ is not in conflict with the conservation of the Kullback distance. This result is consistent with Landauer's assertion that, in classical systems, it is possible to erase a bit against its copy [31, 32] (see [20] for a detailed discussion of this issue in connection with the quantum no deleting theorem). A physical scenario corresponding to the erasure of one bit against its copy can be represented by an initial ensemble of bi-partite source-target systems, such that in each one the source and the target are in the same state (representing a bit and its copy). This ensemble is described by a distribution of the form (37) with $h\left(\boldsymbol{x}^{(s)}, \boldsymbol{x}^{(t)}\right) = \delta\left(\boldsymbol{x}^{(s)} - \boldsymbol{x}^{(t)}\right)$. To briefly discuss the deleting process in terms of Fisher's measure, we consider initial states $\mathcal{P}_\lambda{=}\mathcal{P}^{(m)}_{\text{start}}\left(\boldsymbol{x}^{(m)}\right)\mathcal{P}^{(s)}\left(\boldsymbol{x}^{(s)};\lambda\right)\mathcal{P}^{(s)}\left(\boldsymbol{x}^{(t)};\lambda\right)$ being defined in terms of the mono parametric family of distributions $\mathcal{P}^{(s)}\left(\boldsymbol{x}^{(s)};\lambda\right)$. A universal deleting process would lead to final states of the form $\mathcal{Q}_\lambda{=}\mathcal{P}^{(m)}\left(\boldsymbol{x}^{(m)};\lambda\right)\mathcal{P}^{(s)}\left(\boldsymbol{x}^{(s)};\lambda\right)\mathcal{P}_{\text{blank}}\left(\boldsymbol{x}^{(t)}\right)$. The conservation of Fisher's information, however, implies $I\left[\mathcal{P}^{(s)}(\lambda)\right]{=}I\left[\mathcal{P}^{(m)}(\lambda)\right]$. That is, the Fisher information associated with the initial states of the target systems is entirely transferred to the final states of the machine. One may argue that the "distinguishability" of the initial target states (which is lost during the deleting process) is completely transformed into an equal amount of "distinguishability" of the final state of the machine.

As indicated above, even if Liouville dynamics forbids *universal* cloning or deleting of ensemble distributions, the cloning or deleting of some *particular* distributions are not necessarily forbidden. For instance, if the states to be cloned or deleted are non-overlapping, the Kullback distance is not defined and our present arguments do not hold. Further, in this case the overlap distance between two initial states of the cloning process is equal to the distance between the corresponding final states: both distances vanish. Hence, the conservation of distance is not violated. Similarly, the overlap distances between initial states and between final states of a successful deleting process involving two non-overlapping distributions also vanish. Consequently, the distance can be preserved without transferring any information into the final states of the deleting machine. Entirely known classical states described by δ-distributions are special instances of this "non-overlapping" situation.

From the above considerations we can see that the classical and the quantum no-cloning theorems share an important common feature. In the classical case overlapping probability distributions can not be cloned. In the quantum case, non orthogonal pure states can not be cloned. What classical overlapping probability distributions have in common with non-orthogonal quantum states is that they can not be distinguished with certainty. If we are given one sample x_0 taken from one of two overlapping distributions $p_1(x)$ and $p_2(x)$ we can not tell with certainty which one of the probability distributions $p_{1,2}$ was used to generate x_0. Similarly, if we are given one realization of a quantum system prepared in one of two non-orthogonal states $|\phi_1\rangle$, $|\phi_2\rangle$, there is no way to determine with certainty which of these two states the system was prepared in. Summing up. we can say that both classically and quantum mechanically, non-distinguishable states can not be cloned.

4 Conclusion

Information distances between time dependent solutions of the Liouville equation constitute invariants of the concomitant dynamics. The conservation of these quantities imposes rather strong constraints on possible universal operations in classical ensemble dynamics. These constraints allow for the identification of classical analogues of information-related, quantum mechanical impossible operations such as universal quantum cloning and universal quantum deleting. The Fisher information measure provides an interesting interpretation of these classically forbidden operations in terms of statistical inference theory. The physical impossibility of universal cloning or deleting is a basic feature of classical probabilistic settings arising from an incomplete knowledge of the system's state. However, complete knowledge of classical states is possible, at least in principle, and cloning and deleting are not forbidden in such cases (they are possible even in the more general case of non-overlapping probability distributions). In this regard, the quantum mechanical situation is more strict since universal cloning or deleting are impossible even within the set of completely determined states, that is, for pure states [18, 19, 20]. The present formalism may be applied to investigate classical counterparts of other quantum impossible processes. Possible links between our results and the classical analogue of entanglement analyzed in [13] also deserve further investigation. As a final remarck we want to stress that the existence of classical analogs of the quantum no cloning and related theorems *does not imply* that these quantum impossibility theorems can be completely reduced or understood in terms of classical concepts.

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