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Entropy Bounds, Holographic Principle and Uncertainty Relation

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Abstract: A simple derivation of the bound on entropy is given and the holographic principle is discussed. We estimate the number of quantum states inside space region on the base of uncertainty relation. The result is compared with the Bekenstein formula for entropy bound, which was initially derived from the generalized second law of thermodynamics for black holes. The holographic principle states that the entropy inside a region is bounded by the area of the boundary of that region. This principle can be called the kinematical holographic principle. We argue that it can be derived from the dynamical holographic principle which states that the dynamics of a system in a region should be described by a system which lives on the boundary of the region. This last principle can be valid in general relativity because the ADM hamiltonian reduces to the surface term.

Keywords: Entropy; Uncertainty; Holography; Black hole.

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1 Introduction

There has been a great deal of interest recently in the Bekenstein bound on entropy and the holographic principle as new and perhaps fundamental principles in physics (see [1-20] and Refs. therein). According to Bekenstein [1] there exists a universal bound on the entropy S of any object of maximal radius R and total energy E:

$$S \leq \frac{2\pi}{\hbar c} RE. \tag{1}$$

The bound was derived from the requirement that the generalized second law of thermodynamics for black holes be respected when a box containing entropy is placed without radial motion near the horizon of Schwarzschild black hole and dropped into it [1]. Despite the derivation from gravitation gedanken experiment the bound (1) does not involve the gravitational constant.

The holographic principle [7, 8] states that one has the following bound on the total entropy S contained in a region of space bounded by the spatial surface of the area A,

$$S \leq \frac{A}{4l_p^2}, \tag{2}$$

where l_p is the Planck length $l_p = \sqrt{\hbar G/c^3}$. The bound (2) includes the gravitational constant G.

There are already many discussions of bounds (1) and (2). However these important principles deserve a further study. In this note the number of quantum states inside space region is estimated on the base of uncertainty relation. The result is compared with the Bekenstein formula for entropy bound, which was initially derived from the generalized second law of thermodynamics for black holes. Then we discuss the holographic principle. The holographic principle states that the entropy in a region is bounded by the area of the boundary of the region. This principle can be called the *kinematical* (or thermodynamical) holographic principle. We argue that it can be derived from the *dynamical* holographic principle which states that the dynamics of a system in a region should be described by a system which lives on the boundary of the region. Actually it is well known that the Hamiltonian in general relativity may be reduced to the surface term.

The preliminary version of the present paper is in [18].

Another approach to holography based on chaos is considered in [14, 17].

2 The bound on entropy

In this section we shall give a simple derivation of the bound on entropy. Let us consider a region in 3-dimensional space of characteristic size R, which contains energy E. We use "natural" system of units:

$$c = G = \hbar = 1.$$

Due to the uncertainty relation the minimum energy $\varepsilon(R)$ of particle localized inside the region is of the order $1/R$,

$$\varepsilon(R) \sim 1/R, \tag{3}$$

(since $\varepsilon(R) \sim \sqrt{m_{min}^2 + p_{min}^2} \sim p_{min} \sim 1/R$). The energy $\varepsilon(R)$ can be considered as (minimum) quantum of energy for region with radius R.

The maximum number of particles inside the region for fixed E could be estimated as maximum number of energy quanta, so

$$\mathcal{N}(E,R) \sim \frac{E}{\varepsilon(R)} \sim ER. \tag{4}$$

Let us estimate the maximal entropy of the system.

We have to count the number of quantum states corresponding to given values of E and R. If there is no degeneration of energy levels then our problem is reduced to the counting of number of sets of positive integers $(n_1, \ldots, n_k)$ such that

$$\sum_{i=1}^{k} n_i \leq \mathcal{N}(E,R). \tag{5}$$

Here n_i is the number of energy quanta of the i-th particle. One can easily see that the number of such sets is $2^{\mathcal{N}(E,R)}$. Therefore we obtain for the entropy of a system the bound $S \leq b\mathcal{N}(E,R)$, or, because $\mathcal{N}(E,R) \sim ER$,

$$S \leq bER, \tag{6}$$

where b is a constant. So we have derived the Bekenstein type bound (6) by using basically only the uncertainty relation.

The presence of finite number of internal degrees of freedom and finite number of particle species, as well as the degeneration of energy levels due to the 3-dimensionality of space, does not change the bound (6) (it can change only the constant b, see Appendix).

An analytic argument which supports the bound (6) is given in [19]. The bound (1) for free quantum fields is derived in [20].

3 Kinematical holographic principle

If we assume that the size of the system R with the energy (mass) E is greater than the Schwarzschild radius $2E$, i.e. $2E < R$, then from the bound (6) one gets the bound

$$S \leq \frac{b}{2}R^2, \tag{7}$$

which can be interpreted as the holographic principle [7, 8]. This principle says that the number of quantum degrees of freedom in a region is bounded by the area of the boundary surface. In the such form it can be called the kinematical holographic principle.

4 The Bekenstein bound

The bound (6) is similar to the Bekenstein bound (1). However, this simple derivation does not fix the constant b in (6), which depends on the particle spectrum.

The factor ER does not depend on dimensionality of space-time. Exact calculation in the case of thermal radiation gives the factor $(ER)^{\frac{D-1}{D}}$.

The maximum energy inside the region is of order R (R^{D-3} in D-dimensional case)

$$E_{max}(R) \sim R. \tag{8}$$

For energies above E_{max} the considered region is hidden under horizon, and the consideration of the region from the point of view of distant observer is senseless,

$$S_{max}(R) = \max_E S_{max}(E, R) = S_{max}(E_{max}(R), R) \leq b' R^2. \tag{9}$$

Let us consider the spherical region of radius R, then $E_{max} = R/2$, $S_{max}(R) = \frac{b}{2} R^2$. If we assume that entropy of black hole of radius R ($S_{b.h.} = \pi R^2$) is the maximum entropy of the region of radius R, then it is natural to set $b = 2\pi$, and one gets

$$S_{max}(E, R) \leq 2\pi E R. \tag{10}$$

Formula (10) coincides with (1), nevertheless the problem to calculate the proportionality coefficient b requires a special discussion.

We have to distinguish between derivation of entropy bound by black hole and field theory arguments. The bound derived using black hole arguments represent the maximum entropy of system which can be absorbed by black hole.

For example, let us consider absorbtion of energy E and entropy S by Schwarzschild black hole of mass M. The change of black hole entropy is

$$\delta S_{b.h.} = 4\pi(E^2 + 2ME) \geq S. \tag{11}$$

Relation $\delta S_{b.h.} \geq S$ does not mean that entropy of any system of energy E has to be less than $\delta S_{b.h.}$, actually $\delta S_{b.h.}$ is the maximal entropy of the system of energy E, which can be absorbed by black hole of mass M. If M is small enough S can be greater than $\delta S_{b.h.}$, but in this case absorbtion is impossible.

The bound (1) is independent on mass of black hole, which is supposed to be large in comparison with E, but even this independence does not allow us to conclude that the bound could not be interpreted in similar way. We could imagine that to force system to be absorbed by black hole we have to increase its energy and/or decrease its entropy, or that any box contained the system would be destroyed by pressure of the system. The other possible way is to consider the bound as necessary condition, which has to be valid for any theory compatible with gravity. So, to postulate the bound (1) we have to appeal to some extra arguments besides the gedanken experiments with black holes.

5 Interpretations of the Bekenstein bound

There is the very common implicit assumption: *any system can be absorbed by black hole, if black hole is large enough.* Actually this assumption is not obvious. One can consider for example the gas uniformly distributed in the space such that for its collapse or for the absorbtion by the black hole one needs infinite time.

Let us summarize possible interpretation of bound (1):
(i) $2\pi RE$ is the maximum possible entropy for any selfconsistent theory;
(ii) $2\pi RE$ is the maximum possible entropy for any theory with gravity;
(iii) $2\pi RE$ is the maximum possible entropy for any theory which admits black hole solutions;
(iv) $2\pi RE$ is the maximum possible entropy of system, which can be absorbed by some black hole.

Interpretations (i) and (ii) were mentioned by Bekenstein in [3].

There is no direct observation of black hole, so the interpretation (iii) also can not be skipped without discussion.

Interpretation (iv) was not formulated explicitly, but spirit of this interpretation is present in another Bekenstein paper [2]. It was proved that $2\pi RE$ becomes maximal entropy of the system of large number of particle species only if we take into account energy of the box restricted the system, i.e. system with entropy larger then $2\pi RE$ could be absorbed by black hole only inside the box, which energy is large enough to satisfy Bekenstein bound for whole system. This example demonstrates the fashion of trick, which could be used by nature in the case of interpretation (iv) to protect second law of thermodynamics. In special relativity there is no sense a notion of perfect solid body as well as a notion of a massless box in quantum field theory. We do not know which abstractions will be obsolete in quantum gravity. Perhaps we have to take into account an energy exchange between the system in the box and space-time. It maybe also that a very high potential barrier around the internal space of the box is an unacceptable approximation.

An alternative bound was also suggested by Unruh and Wald [5, 6],

$$S \leq Vs(E/V). \qquad (12)$$

According to the Bekenstein paper [2] the Unruh and Wald bound (12) introduced in [5, 6] is neither necessary, nor sufficient. Nevertheless it is natural to assume, that thermal radiation is maximally entropic in the classical case (low energy, intermediate volume without gravity effects).

It is the well known fact (see [4]), that for $V < V_{cr}$,

$$V_{cr} = const E^5, \tag{13}$$

thermal radiation becomes unstable and it partially collapses into black hole. For a given volume and energy, if $V > V_{cr}$ then we can expect that the system with only thermal radiation will have the maximum entropy. In this case the Unruh and Wald bound (12) is valid. However in the case $V < V_{cr}$, the system should have the maximum entropy if it contains the combination of thermal radiation and black hole. Bekenstein bound probably covers both ranges of V.

6 Dynamical holographic principle

One may formulate also the principle that the dynamics of a physical system in a region which includes gravity should be described by a system which lives on the boundary of the region. Such principle can be called the dynamical holographic principle. It is discussed in the context of the AdS/CFT correspondence [10, 11, 12, 13, 15, 16].

We would like to point out that the dynamical holographic principle in a certain sense is valid in classical general relativity. This is due to the well known fact that the density of the ADM Hamiltonian $H(x)$ in general relativity is the total divergence. One has (see e.g. [24])

$$H(x) = T_0(x) - \partial_i \partial_k q^{ik}, \tag{14}$$

where

$$T_0(x) = q^{ij} q^{kl} \left(\pi_{ik} \pi_{jl} - \pi_{lj} \pi_{kl} \right) + g_3 R_3 \tag{15}$$

with

$$q^{ik} = h^{i0} h^{k0} - h^{00} h^{ik}, \quad h^{\mu\nu} = \sqrt{-g} g^{\mu\nu}. \tag{16}$$

By the constraint

$$T_0 = 0 \tag{17}$$

the Hamiltonian reduces to the surface term

$$H = \frac{1}{2} \int H(x) d^3x = -\frac{1}{2} \int \partial_i \partial_k q^{ik} d^3x = -\frac{1}{2} \int \partial_k q^{ik} d\sigma_i. \tag{18}$$

This means that all dynamical information is encoded in the data on the boundary surface. Of course the system is highly nonlinear because we have to solve constraints but in principle one has the dynamical holography in classical general relativity.

One gets the similar conclusion if one uses the pseudo-tensor [25] which leads to the expression for the energy-momentum as the integral over the boundary surface

$$P^{\mu} = \int h^{\mu\nu\lambda} d\sigma_{\nu\lambda}, \tag{19}$$

where

$$h^{\mu\nu\lambda} = \frac{1}{2}\partial_{\sigma}\left((-g)(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma})\right). \tag{20}$$

Now by having the classical dynamical holographic principle we can argue for the quantum kinematical principle. In fact, if the energy and other dynamical quantities of the system are expressed as the integrals over the boundary surface then the number of quantum states of the system should be estimated by the area of the surface. We have to assume a cutoff on the surface to remove divergences.

7 Conclusion

In this note the bounds on entropy and the holographic principle have been discussed. By using the simple counting arguments and uncertainty relation we have demonstrated that a general Bekenstein type bound (6) is valid. It is pointed out also that one has to distinguish between the kinematical and dynamical holographic principle and that the last one is actually valid in classical general relativity. We have argued that the quantum kinematical holographic principle can be derived from the classical holography. However, it is an open question whether this classical holography can be used for the rigorous justification of the quantum holographic principle because of the problem of divergences.

Appendix

It is convenient to use the notion of entropy as deficit of information we need to describe the system state.

Let us assume that we have to take into account only finite number of particle species, each sort of particle has finite number of internal states. This assumtion seems to be natural for large R (i.e. for small ε). So we need only finite number of bits of information to describe internal degrees of freedom of one particle. The maximal number of particles is about $\mathcal{N}(E,R)$, so we need about $\mathcal{N}(E,R)$ bits of information.

Due to uncertainty principle we can determine momentum with accuracy of $\varepsilon(R)$, so x projection of momentum of the i-th particle can be described by natural n_i^x and sign "+" or "−". We can describe these signs using $\mathcal{N}(E,R)$ bits of information. One has

$$p_i^x = \pm n_i^x \varepsilon(R), \tag{21}$$

$$\begin{aligned} E_i &\geq |p_i^x|, && (22)\\ E &= \mathcal{N}(E,R)\varepsilon(R) = \sum_i E_i \geq \sum_i n_i^x \varepsilon(R) && (23) \end{aligned}$$

so

$$\mathcal{N}(E,R) \geq \sum_i n_i^x. \qquad (24)$$

To count the number of possible sets of n_i^x, which satisfy relation (24) we consider the following string

$$(1w1w1w\ldots w1w1w0)\cdot 0, \qquad (25)$$

which contain $\mathcal{N}(E,R)$ symbols "1" and $\mathcal{N}(E,R)$ symbols "w". To describe any possible sum of the form (24) we have to replace all "w" by the following strings: "+" or ")+(". We can use just $3\cdot 2\cdot \mathcal{N}(E,R)$ bits to describe momenta.

Finally, for large R we get

$$S_{max}(E,R) \leq bER. \qquad (26)$$

where the constant b does not depend on E and R.

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