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# Some Divergence Properties of Asset Price Models

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**Abstract:** We consider asset price processes $X_t$ which are weak solutions of one-dimensional stochastic differential equations of the form

$$dX_t \ = \ b(t, X_t) \ dt \ + \ \sigma_t \ X_t \ dW_t.$$

Such price models can be interpreted as non–lognormally–distributed generalizations of the geometric Brownian motion. We study properties of the $I_\alpha$–divergence between the law of the solution $X_t$ and the corresponding drift–less measure (the special case $\alpha = 1$ is the relative entropy). This will be applied to some context in statistical information theory as well as to arbitrage theory and contingent claim valuation. For instance, the seminal option pricing theorems of Black-Scholes and Merton appear as a special case.

**Keywords:** $I_\alpha$–divergence; relative entropy; statistical information; equivalent martingale measure; option pricing; Black-Scholes-Merton.

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## 1 Introduction

One popular model for the time-behaviour of financial asset prices is the geometric Brownian motion $X_t$, which is the unique (strong) solution of the stochastic differential equation (SDE)

$$dX_t \; = \; C_1 \; X_t \; dt \; + \; \sigma^{co} \; X_t \; dW_t \; , \tag{1}$$

with some constants $C_1 \in I\!R$ and $\sigma^{co} > 0$, and Brownian motion $W$ (see Samuelson [25], Merton [17], and Merton [18]). This model was also used by Black & Scholes [1] and Merton [19] for the underlying in their seminal investigations on option pricing. Since the geometric Brownian motion model has some deficiences, it makes sense to study asset price processes which are weak solutions of the SDE

$$dX_t \; = \; b(t, X_t) \; dt \; + \; \sigma_t \; X_t \; dW_t \quad , \qquad t \in [s, T] \; , \tag{2}$$
$$X_s \; \textit{has probability distribution} \; \mu \; \textit{on} \; ]0, \infty[ \; ,$$

where $s \in [0, T]$ is the starting time, $\mu$ *on* $]0, \infty[$ is the starting distribution, and $T > 0$ is the fixed final time horizon. The corresponding solution measure will be denoted by $Q_{(s,\mu)}$, and the solution measure for the case $b(t, X_t) \equiv 0$ (with some different Brownian motion $\overline{W}$) will be denoted by $P_{(s,\mu)}$. The underlying measurable space and filtrations will be mostly omitted for the sake of brevity.

The SDE (2) can be thought of as a "generalized Samuelson-Merton-Black-Scholes world"; its solutions $(X_t, Q_{(s,\mu)})$ are — in contrast to the geometric Brownian motion — typically non-lognormally distributed.

Furthermore, we deal with the (for our context adapted form of the) "$I_\alpha-$divergences"

$$I_\alpha(Q_{(s,\mu)} || P_{(s,\mu)}) \; := \; \int \; f_\alpha\left(\frac{dQ_{(s,\mu)}}{dP_{(s,\mu)}}\right) \; dP_{(s,\mu)} \; ,$$

with the nonnegative functions $f_\alpha : [0, \infty[ \to [0, \infty[$ defined by

$$f_\alpha(\rho) := \begin{cases} -\log \rho + \rho - 1, & \textit{if} \quad \alpha = 0, \\ \frac{\alpha\rho + 1 - \alpha - \rho^\alpha}{\alpha(1-\alpha)}, & \textit{if} \quad \alpha \in I\!R \backslash \{0, 1\}, \\ \rho \; \log \rho + 1 - \rho, & \textit{if} \quad \alpha = 1; \end{cases}$$

for investigations on $I_\alpha-$divergences for general measures, see e.g. Liese & Vajda [13]. As usual, one makes the conventions $-\log 0 = \infty$, and $0 \; \log 0 = 0$. The case $\alpha = 1$ corresponds to the relative entropy; some other prominent special cases are the (double) Hellinger distance ($\alpha = \frac{1}{2}$)

and (half of) the $\chi^2-$divergence ($\alpha = 2$). Clearly, the integrals in $I_\alpha$ always exist (if the involved density exists); however, it is well known that, for general measures, one can only guarantee the finiteness of the $I_\alpha-$divergences for $\alpha \in ]0,1[$ (i.e. the relative entropy and the $\chi^2-$divergence are not covered).

The main goals of this paper are :

- (1) to present some assumptions (e.g. of stochastic, exponential type), such that the divergences $I_\alpha(Q_{(s,\mu)}||P_{(s,\mu)})$ are finite for a parameter range which is strictly larger than $]0,1[$. We also give upper bounds for a certain worst–case scenario. For the special case of time-homogeneous drifts, we also investigate the corresponding short-time and long-time behaviour. (see Section 2).
- (2) to apply the results of (1) in order to deduce assertions upon the statistical information contained in two particular dichotomous decision problems (see Section 3). Another application deals with a "direct" method to verify the existence of a unique equivalent martingale measure as well as with the corresponding no-arbitrage result and the corresponding unique arbitrage-based prices of European contingent claims on the underlying instrument $X$. (see Section 4).

For the sake of a smoother presentation, the proofs will be given in the final Section 5.

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## 2 Divergence results

In general, we always suppose that the drift $b : [0,T] \times [0,\infty[ \mapsto [-\infty,\infty]$ and the volatility function $\sigma : [0,T] \mapsto [0,\infty]$ are deterministic, (possibly exploding) Borel measurable functions. Furthermore, for the special case with the Dirac-measure $\vartheta_x$ as the starting distribution (i.e., the process $X$ starts in $x > 0$) we will write $Q_{(s,x)}$ instead of $Q_{(s,\vartheta_x)}$, etc.

As a first set of further assumptions, we shall use the following:

**Assumption 2.1** *The deterministic volatility function $\sigma_t$ satisfies the condition*

$$\int_0^T \sigma_v^2 \, dv \; < \; \infty \; . \tag{3}$$

**Assumption 2.2** *One of the following 6 conditions holds (where $EP_{(u,x)}$ denotes the expectation with respect to the drift-less measure $P_{(u,x)}$):*

$$(a) \qquad \sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)}\Big[ \, exp\Big(\frac{1}{2}\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big) \, \Big] \; < \; \infty \, . \tag{4}$$

$$(b) \qquad \exists n \in I\!N : \; \frac{1}{2^n \; n!} \sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)}\Big[ \, \Big( \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big)^n \, \Big] \; < \; 1 \, .$$

$$(c) \qquad \lim_{n\to\infty} \Big\{ \, \frac{1}{2^n \; n!} \sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)}\Big[ \, \Big( \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big)^n \, \Big] \, \Big\}^{\frac{1}{n}} \; < \; 1 \, .$$

$$(d) \qquad \lim_{n\to\infty} \Big\{ \, \frac{1}{2^n \; n!} \sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)}\Big[ \, \Big( \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big)^n \, \Big] \, \Big\} \; = \; 0 \, .$$

$$(e) \qquad \sum_{n=0}^{\infty} \frac{1}{2^n \; n!} \Big\{ \sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)}\Big[ \, \Big( \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big)^n \, \Big] \, \Big\} \; < \; \infty \, .$$

$$(f) \qquad \sup_{(u,x)\in[0,T]\times]0,\infty[} \Big\{ \, \sum_{n=0}^{\infty} \frac{1}{2^n \; n!} EP_{(u,x)}\Big[ \, \Big( \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big)^n \, \Big] \, \Big\} \; < \; \infty \, .$$

**Remarks:** (R1) For the case of constant volatility function $\sigma_t \equiv \sigma^{co}$, these 6 conditions can be interpreted as worst–case scenarios (of integrals) with respect to the drift–less geometric Brownian motion (i.e. $C_1 = 0$ in (1)).
(R2) The condition (4) is a uniform version of the corresponding Novikov condition.

With the help of the above assumptions, one gets the following assertions associated with the existence of a solution of (2) :

**Theorem 2.3** *Suppose the Assumptions 2.1 and 2.2. Then*
*(a) for all starting times $s \in [0,T]$ and all starting distributions $\mu$ on $]0,\infty[$, one has the Girsanov condition*

$$EP_{(s,\mu)}\Big[ \; Z_{s,T} \; \Big] \; = \; 1 \; , \tag{5}$$

*where*

$$Z_{s,t} \;:=\; exp\Big( \int_s^t \frac{b(v,X_v)}{\sigma_v \; X_v} \, d\overline{W}_v \;-\; \frac{1}{2} \int_s^t \frac{b(v,X_v)^2}{\sigma_v^2 \; (X_v)^2} \, dv \Big) \;, \qquad t \in [s,T] \;.$$

*(b) for all starting times* $s \in [0,T]$ *and all starting distributions* $\mu$ *on* $]0,\infty[$, *a weak solution of the SDE (2) is given by the triple* $(X, W, Q_{(s,\mu)})$, *with (i)* $Q_{(s,\mu)} := Z_{s,T} * P_{(s,\mu)}$ *having density* $Z_{s,T}$ *with respect to* $P_{(s,\mu)}$, *and (ii)* $W_t := \overline{W}_t - \overline{W}_s - \int_s^t \frac{b(v,X_v)}{\sigma_v \; X_v} \, dv$, $t \in [s,T]$. *This solution has the property, that for all* $t \in [0,T]$ *there holds* $X_t > 0$ ($Q_{(s,\mu)}-$*almost surely).*
*(c) within the class of weak solutions* $\Big(\widehat{X}, \widehat{W}, \widehat{Q_{(s,\mu)}}\Big)$ *of (2) with the additional property*

$$\widehat{Q_{(s,\mu)}}\Big[ \int_s^T \frac{b(v,\widehat{X_v})^2}{\sigma_v^2 \; \widehat{X_v}^2} \, dv \;<\; \infty \Big] = 1,$$

*the solution* $(X, W, Q_{(s,\mu)})$ *is unique (in law).*

We are now in the position to present some divergence properties:

**Theorem 2.4** *Suppose the Assumptions 2.1 and 2.2. Then the following assertions hold :*
*(a) for all* $\alpha \in \, ]\inf_{\epsilon \in \mathcal{U}} h_1(\epsilon), \, \sup_{\epsilon \in \mathcal{U}} h_2(\epsilon)[$, *all starting times* $s \in [0,T]$, *and all probability measures* $\mu$ *on* $]0,\infty[$ *:*

$$0 \;\leq\; I_\alpha(Q_{(s,\mu)} || P_{(s,\mu)}) \;<\; \infty \;,$$

*where we define the three quantities*

$$\mathcal{U} := \Big\{ \epsilon > 0 \;\Big|\; \sup_{(u,x) \in [0,T] \times ]0,\infty[} EP_{(u,x)} \Big[ \, exp\Big( \frac{1+\epsilon}{2} \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2 \; (X_v)^2} \, dv \Big) \Big] \;<\; \infty \Big\} \quad ,$$

$$h_1(\epsilon) \;:=\; - \, \frac{1+\epsilon}{2\sqrt{1+\epsilon}+1} \quad , \quad and \quad h_2(\epsilon) \;:=\; \frac{1+\epsilon}{2\sqrt{1+\epsilon}-1} \quad .$$

*(b) according to the value of* $\alpha$, *an upper bound for the quantity* $\sup_{(u,x) \in [0,T] \times ]0,\infty[} \; I_\alpha(Q_{(u,x)} || P_{(u,x)})$ *is :*

$$(i) \quad \frac{1}{2} \sup_{(u,x) \in [0,T] \times ]0,\infty[} EP_{(u,x)} \Big[ \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2 \; (X_v)^2} \, dv \Big] \;, \qquad\qquad if \quad \alpha = 0.$$

$$(ii) \quad \frac{1}{\alpha(1-\alpha)} \Big\{ 1 - \Big( \sup_{(u,x) \in [0,T] \times ]0,\infty[} EP_{(u,x)} \Big[ \, exp\Big( \frac{\alpha^2 (1 + \sqrt{1 - \frac{1}{\alpha}})^2}{2} \;\times$$

$$\times \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\ (X_v)^2}\ dv\Big)\Big]\Big)^{\frac{\sqrt{1-\frac{1}{\alpha}}}{1+\sqrt{1-\frac{1}{\alpha}}}}\Bigg\}\ ,$$

$$if \quad \alpha \in \ ]\inf_{\epsilon\in\mathcal{U}} h_1(\epsilon), 0[\ \cup\ ]1, \sup_{\epsilon\in\mathcal{U}} h_2(\epsilon)[.$$

$$(iii) \qquad \frac{1}{\alpha(1-\alpha)}\Bigg\{1-$$

$$\frac{1}{\Big(\sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)}\Big[exp\Big(\frac{1}{2}\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\ (X_v)^2}\ dv\Big)\Big]\Big)^{\frac{\alpha+1-\sqrt{1-2\alpha+5\alpha^2}}{2}}}\Bigg\}\ ,$$

$$if \quad \alpha \in \ ]0,1[.$$

$$(iv) \qquad \inf_{q_1\in]1,\sup_{\epsilon\in\mathcal{U}} h_2(\epsilon)[} \Bigg(\sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)}\Big[\ exp\Big(\frac{(q_1)^2\ (1+\sqrt{1-\frac{1}{q_1}})^2}{2}$$

$$\times \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\ (X_v)^2}\ dv\Big)\Big]\Bigg)^{\frac{\sqrt{1-\frac{1}{q_1}}}{q_1+q_1\sqrt{1-\frac{1}{q_1}}}}$$

$$\times\ \frac{1}{2}\ \Bigg(\sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)}\Big[\ \Big(\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\ (X_v)^2}\ dv\Big)^{\frac{q_1}{q_1-1}}\Big]\ \Bigg)^{\frac{q_1-1}{q_1}}\ , \qquad if \quad \alpha = 1.$$

**Remarks:** (R1) The set $\mathcal{U}$ is always non–empty (see the proof in Section 5).
(R2) The function $h_1(\cdot)$ is strictly decreasing, and one has $\lim_{\epsilon\downarrow 0} h_1(\epsilon) = -\frac{1}{3}$.
(R3) The function $h_2(\cdot)$ is strictly increasing, and one has $\lim_{\epsilon\downarrow 0} h_2(\epsilon) = 1$.
(R4) As a continuation of the parameter discussion in Section 1: because of the first three remarks, our interval of all admissible $\alpha$ (i.e. for which the $I_\alpha-$divergence is finite) is strictly larger (on both sides) than $[-\frac{1}{3}, 1]$. In particular, the relative entropy ($\alpha = 1$) is covered.
(R5) For the case $\alpha \in\ ]0,1[$, one knows in the general theory of $I_\alpha-$divergences only the larger upper bound $\frac{1}{\alpha(1-\alpha)}$. For our particular set-up, one gets the improved, smaller upper bound (biii).
(R6) The upper bounds in (b) can be interpreted as worst-case-scenario estimates.

What happens when we assume the "ordinary" Novikov condition (which also guarantees the existence of a weak solution of (2)) instead of the uniform version (4)?

**Assumption 2.5** *For fixed probability measure $\widehat{\mu}$ on $]0,\infty[$ and fixed starting time $s \in [0,T]$, the Novikov condition*

$$EP_{(s,\widehat{\mu})}\Big[\ exp\Big(\frac{1}{2}\int_s^T \frac{b(v,X_v)^2}{\sigma_v^2\ (X_v)^2}\ dv\Big)\Big]\ <\ \infty$$

*holds.*

In this case, one can only deduce the following weaker divergence properties:

**Theorem 2.6** *Suppose the Assumptions 2.1 and 2.5. Then the following assertions hold :*
*(a) for all $\alpha \in\ [-\frac{1}{3},1[$ :*

$$0\ \leq\ I_\alpha(Q_{(s,\widehat{\mu})}||P_{(s,\widehat{\mu})})\ <\ \infty\ .$$

*(b) according to the value of $\alpha$, an upper bound for the quantity $I_\alpha(Q_{(s,\widehat{\mu})}||P_{(s,\widehat{\mu})})$ is given by the same expression as in Theorem 2.4(b), with replacing the term "$\sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)}$" by "$EP_{(s,\widehat{\mu})}$" and the term "$\int_u^T$" by "$\int_s^T$".*

**Remark:** Notice that the relative entropy ($\alpha = 1$) is not covered in Theorem 2.6.

For the special case of time-homogeneous drifts $b(X_v)$ (but still with non-constant volatility function), we use the following assumptions:

**Assumption 2.7** *The volatility function $\sigma_t$ satisfies the two conditions (3) and*

$$\lim_{\Delta t\downarrow 0}\ \sup_{u\in[0,T]}\ \int_u^{(u+\Delta t)\wedge T} \frac{1}{\sigma_v^2\ \sqrt{\int_u^v \sigma_{v_1}^2\ dv_1}}\ dv\ =\ 0\ . \tag{6}$$

**Assumption 2.8** *Either (i) one of the 6 stochastic conditions (a) to (f) of Assumption 2.2 (with $b(X_v)$ instead of $b(v,X_v)$),*
*or (ii) the non-stochastic condition*

$$\sup_{a\in\mathbb{R}} \int_{a-1}^{a+1} \frac{\{b(e^\varsigma)\}^2}{e^{2\varsigma}}\ d\varsigma\ <\ \infty\ , \tag{7}$$

*holds.*

In contrast to the case of time-inhomogeneous drifts $b(v,X_v)$, one gets much nicer divergence properties:

**Theorem 2.9** *Suppose the Assumptions 2.7 and 2.8. Then the following assertions hold :*
*(a) for all $\alpha \in I\!R$, all $s \in [0,T]$, and all probability measures $\mu$ on $]0,\infty[$ :*

$$0 \leq I_\alpha\Big(Q_{(s,\mu)}\Big|\Big|P_{(s,\mu)}\Big) < \infty .$$

*(b) for all $\alpha \in I\!R$ :*

$$\lim_{\Delta t \downarrow 0} \sup_{(u,x)\in[0,T]\times]0,\infty[} I_\alpha\Big(Q_{(u,x)}^{(u+\Delta t)\wedge T}\Big|\Big|P_{(u,x)}^{(u+\Delta t)\wedge T}\Big) = 0 , \tag{8}$$

*where $Q_{(u,x)}^{(u+\Delta t)\wedge T}$ denotes the restriction of $Q_{(u,x)}$ to the time interval $[u,(u+\Delta t)\wedge T]$ (i.e. the process $X$ starts at time $u$ and is observed until the time $(u+\Delta t)\wedge T$).*
*(c) according to the value of $\alpha$, for any fixed starting time $u \in [0,T]$ and any starting point $x > 0$ the time evolution (with respect to $\Delta t > 0$) of the divergence $I_\alpha\Big(Q_{(u,x)}^{(u+\Delta t)\wedge T}\Big|\Big|P_{(u,x)}^{(u+\Delta t)\wedge T}\Big)$ can be estimated from above by the following function $h_3(\Delta t) :=$*

$$\begin{array}{ll}
c_1 + c_2\,\Delta t , & if \quad \alpha = 0, \\[2ex]
\dfrac{1}{\alpha(1-\alpha)}\Big\{1 - exp\Big(c_3(\alpha) + c_4(\alpha)\ \Delta t\Big)\Big\} , & if \quad \alpha \in I\!R\backslash[0,1], \\[2ex]
\dfrac{1}{\alpha(1-\alpha)}\Big\{1 - exp\Big(-c_5(\alpha) - c_6(\alpha)\ \Delta t\Big)\Big\} , & if \quad \alpha \in\, ]0,1[, \\[2ex]
\Big\{c_7 + c_8\ \Delta t\Big\}\ exp\Big(c_9 + c_{10}\ \Delta t\Big) , & if \quad \alpha = 1,
\end{array} \tag{9}$$

*with some* **strictly** *positive constants $c_1, c_2, c_3(\alpha), c_4(\alpha), c_5(\alpha), c_6(\alpha), c_7, c_8, c_9,$ and $c_{10}$; these constants depend on the drift $b$, the volatility function $\sigma_t$ and, as far as indicated, also on the parameter $\alpha$. All these constants are independent of the starting time $u$, the starting point $x$ and the evolution time $\Delta t$.*

**Remarks:** (R1) Notice that, in general, with the assumptions of Theorem 2.9 one can only guarantee the **strict** positiveness of all constants in (9), in particular of $c_1, c_3(\alpha), c_5(\alpha)$ and $c_7$. Therefore, the condition (8) can not be derived from (c) by taking $\lim_{\Delta t\downarrow 0} h_3(\Delta t)$ in (9). For instance, in the case $\alpha = 0$ this limit is $c_1 > 0$. However, there is of course no contradiction since (9) has to cover the time-evolution also for large $\Delta t$.
(R2) As a continuation of the parameter discussion in Remark (R4) after Theorem 2.4: the interval of all admissible $\alpha$ is now the whole real line! In particular, the $\chi^2-$divergence ($\alpha = 2$) is also

covered.
(R3) The assertion (b) in Theorem 2.9 can be interpreted as the short time-behaviour for a worst-case scenario; in contrast, the assertion (c) also covers the long-time behaviour.
(R4) For the special case of supposing the Assumption 2.7 together with the non-stochastic condition (7), see also Stummer [26], where one can find examples, too.

## 3 Applications to information theory

For general measures, Österreicher & Vajda [22] have given some nice links between $I_\alpha-$divergences and statistical information measures of some dichotomous decision problems. In this section, we adapt those results to our special asset price model set-up, and use them in combination with our investigations of Section 2:

Imagine that, prior to time $s \geq 0$, you have become interested in an asset $X$, and that you have reasons to model the future dynamics of the corresponding price process $X_t$ in the time interval $[s,T]$ by a stochastic differential equation of the form (2). Furthermore, suppose that you have fixed (i) the starting distribution $\mu$ at time $s$, and (ii) the deterministic volatility function $\sigma_t$.

However, assume that you don't know whether $X$ has a certain drift $b$ or it has zero drift, and therefore you want to decide, in an optimal Bayesian way, which **degree of evidence** $\gamma$ you should attribute (according to a pregiven **loss function** $\mathcal{L}$) to the "event" that $X$ has drift $b$.

In order to achieve this goal, you choose a loss function $\mathcal{L}(\theta,\gamma)$ defined on $\{0,1\} \times [0,1]$. Furthermore, according to your beliefs (or experiments) prior to time $s$, you fix a **prior (binomial) probability** $p \in \,]0,1[$ for the event $\theta = 1$, which is associated with the drift-bearing measure $Q_{(s,\mu)}$. Also, you attach the prior (binomial) probability $1-p$ to the event $\theta = 0$, which is associated with the drift–less measure $P_{(s,\mu)}$. It is assumed that the prior probability $p$ should not depend on $s$ and $\mu$.

The **risk** (or uncertainty), **prior** to time $s$, from the optimal decision about the degree of evidence $\gamma$ concerning the **decision parameter** $\theta$, is defined as

$$\mathcal{BR}_{\mathcal{L}}(p) := \inf_{\gamma\in[0,1]} \Big\{ \; (1-p)\;\mathcal{L}(0,\gamma) \;+\; p\;\mathcal{L}(1,\gamma) \; \Big\} \; .$$

In order to reduce the decision risk, imagine that you plan to continuously observe a path of $X_t$ in the future time interval $[s,T]$. The corresponding risk (or uncertainty), **posterior** to the observation of $X$, from the optimal decision about the degree of evidence $\gamma$ concerning the parameter $\theta$, is given by

$$\mathcal{BR}_{\mathcal{L}}(p \ ||| \ Q_{(s,\mu)}, P_{(s,\mu)}) := \int_{\Omega} \mathcal{BR}_{\mathcal{L}}(p_{post}) \ (p \ dQ_{(s,\mu)} \ + \ (1-p) \ dP_{(s,\mu)}) \ ,$$

with **posterior probabilities** $p_{post} := \frac{p \ Z_{s,T}}{p \ Z_{s,T} \ + \ (1-p)}$. The **statistical information measure** (in the sense of De Groot [3])

$$\Delta\mathcal{BR}_{\mathcal{L}}(p, Q_{(s,\mu)}, P_{(s,\mu)}) := \mathcal{BR}_{\mathcal{L}}(p) - \mathcal{BR}_{\mathcal{L}}(p \ ||| \ Q_{(s,\mu)}, P_{(s,\mu)})$$

represents the reduction of the decision risk about the degree of evidence $\gamma$ concerning the parameter $\theta$, that can be attained by observing the path of $X$ in the time interval $[s, T]$.

Naturally, it makes sense to ask the following question: How much is the (average) statistical information which is contained in the above-mentioned dichotomous Bayesian decision problem?

Clearly, the answer to this question depends essentially on the choice of the loss function $\mathcal{L}$. Let us first investigate the following

**Context 1.** Consider the loss function $\mathcal{L}_0(\theta, \gamma) := \gamma - (2 \ \gamma - 1) \ \mathbf{1}_{\{1\}}(\theta)$, defined on $\{0,1\} \times [0,1]$, where $\mathbf{1}_A(.)$ denotes the indicator function on a set $A$. This corresponds to the Bayesian **testing** problem $H_0 : Q_{(s,\mu)}$ against the drift–less alternative $H_1 : P_{(s,\mu)}$ (since – because of the specific form of $\mathcal{L}_0$ – one always ends up with deciding only between the extremal evidence degrees $\gamma = 1$ or $\gamma = 0$). For this situation, Österreicher & Vajda [22] obtained the representation formula (which we have adapted to our context of asset price processes)

$$I_\alpha(Q_{(s,\mu)}||P_{(s,\mu)}) \ = \ \int_0^1 \Delta\mathcal{BR}_0(p, Q_{(s,\mu)}, P_{(s,\mu)}) \ dF_\alpha(p) \ , \tag{10}$$

with $dF_\alpha(p) = (1-p)^{\alpha-2} \ p^{-1-\alpha} \ dp$. For the parameter range, we use all $\alpha$ for which $I_\alpha$ is finite. Here and henceforth, the abbreviation $\Delta\mathcal{BR}_0 := \Delta\mathcal{BR}_{\mathcal{L}_0}$ is used.

By combining (10) with the Theorems 2.4 and 2.6, one obtains the following statements on the $F_\alpha$–weighted–average statistical information measure for the abovementioned testing problem:

**Corollary 3.1** *Suppose the Assumption 2.1.*
*(a) If the Assumption 2.2 is satisfied, then the following assertions hold :*
*(i) for all* $\alpha \in \, ]\inf_{\epsilon \in \mathcal{U}} h_1(\epsilon), \ \sup_{\epsilon \in \mathcal{U}} h_2(\epsilon)[$, *all starting times* $s \in [0, T]$, *and all probability measures* $\mu$ *on* $]0, \infty[$, *one has*

$$0 \ \leq \ \int_0^1 \Delta\mathcal{BR}_0(p, Q_{(s,\mu)}, P_{(s,\mu)}) \ \frac{(1-p)^{\alpha-2}}{p^{\alpha+1}} \ dp \ < \ \infty \ . \tag{11}$$

*(ii) according to the value of $\alpha$, an upper bound for the quantity*

$$\sup_{(u,x)\in[0,T]\times]0,\infty[} \int_0^1 \Delta\mathcal{BR}_0(p, Q_{(u,x)}, P_{(u,x)}) \, \frac{(1-p)^{\alpha-2}}{p^{\alpha+1}} \, dp$$

*is given by the estimate in Theorem 2.4(b).*
*(b) If the Assumption 2.5 is satisfied, then the following assertions hold :*
*(i) for all $\alpha \in [-\frac{1}{3}, 1[$, the inequalities (11) are satisfied (with the fixed starting time $s \in [0,T]$ and the fixed starting distribution $\widehat{\mu}$ instead of $\mu$).*
*(ii) according to the value of $\alpha$, an upper bound for the quantity $\int_0^1 \Delta\mathcal{BR}_0(p, Q_{(s,\widehat{\mu})}, P_{(s,\widehat{\mu})}) \frac{(1-p)^{\alpha-2}}{p^{\alpha+1}} dp$ is given by the estimate described in Theorem 2.6(b).*

In contrast to Context 1, let us now deal with a different kind of loss function:

**Context 2.** Consider $\mathcal{L}_{\alpha,\zeta}(\theta,\gamma) := \frac{\alpha^{\theta-1} \;\; \gamma^{\alpha-\theta}}{\zeta^{\alpha} \;\; (1-\zeta)^{1-\alpha} \;\; (1-\alpha)^{\theta} \;\; (1-\gamma)^{\alpha-\theta}}$ which is defined on $\{0,1\} \times [0,1]$, with parameters $\alpha \in ]0,1[$ and $\zeta \in ]0,1[$. For this situation, Österreicher & Vajda [22] obtained the representation formula (which we have adapted to our framework of asset price processes)

$$I_\alpha(Q_{(s,\mu)}||P_{(s,\mu)}) \;=\; \Delta\mathcal{BR}_{\alpha,p}(p, Q_{(s,\mu)}, P_{(s,\mu)}) \; . \tag{12}$$

Here and henceforth, we abbreviate $\Delta\mathcal{BR}_{\alpha,p} := \Delta\mathcal{BR}_{\mathcal{L}_{\alpha,p}}$. If one now combines (12) with the Theorems 2.4 and 2.6, one gets the following estimates on the statistical information measure which is associated with the dichotomous Bayesian decision problem with loss function $\mathcal{L}_{\alpha,\zeta}$:

**Corollary 3.2** *Suppose the Assumption 2.1.*
*(a) If the Assumption 2.2 is satisfied, then – according to the value of $\alpha \in ]0,1[$ – for all prior probabilities $p \in ]0,1[$ an upper bound for the quantity*

$$\sup_{(u,x)\in[0,T]\times]0,\infty[} \Delta\mathcal{BR}_{\alpha,p}\Big(p, Q_{(u,x)}, P_{(u,x)}\Big)$$

*is given by the estimate in Theorem 2.4(b); notice that this estimate does not depend on $p \in ]0,1[$.*
*(b) If the Assumption 2.5 is satisfied, then – according to the value of $\alpha \in ]0,1[$ – for all prior probabilities $p \in ]0,1[$ an upper bound for the quantity $\Delta\mathcal{BR}_{\alpha,p}(p, Q_{(s,\widehat{\mu})}, P_{(s,\widehat{\mu})})$ is given by the estimate described in Theorem 2.6(b); notice that this estimate does not depend on $p \in ]0,1[$.*

For the important special case of time-homogeneous drifts $b(t, X_t) = b(X_t)$ (but still with time–dependent volatility function $\sigma_t$), one gets – with the help of Theorem 2.9 – nicer types of results on the statistical information measures described above:

**Corollary 3.3** *Suppose the Assumptions 2.7 and 2.8. Then the following assertions hold :*
*(a) for all* $\alpha \in I\!R$*, all* $s \in [0,T]$*, and all probability measures* $\mu$ *on* $]0,\infty[$*, the finiteness condition (11) is satisfied.*
*(b) for all* $\alpha \in I\!R$ *:*

$$\lim_{\Delta t \downarrow 0} \sup_{(u,x)\in[0,T]\times]0,\infty[} \int_0^1 \Delta\mathcal{BR}_0\Big(p, Q_{(u,x)}^{(u+\Delta t)\wedge T}, P_{(u,x)}^{(u+\Delta t)\wedge T}\Big) \, \frac{(1-p)^{\alpha-2}}{p^{\alpha+1}} \, dp \;=\; 0.$$

*(c) according to the value of* $\alpha$*, for any fixed starting time* $u \in [0,T]$ *and any starting point* $x > 0$*, the time evolution (with respect to* $\Delta t > 0$*) of the average statistical information measure*

$$\int_0^1 \Delta\mathcal{BR}_0\Big(p, Q_{(u,x)}^{(u+\Delta t)\wedge T}, P_{(u,x)}^{(u+\Delta t)\wedge T}\Big) \, \frac{(1-p)^{\alpha-2}}{p^{\alpha+1}} \, dp$$

*can be estimated from above by the function* $h_3(\Delta t)$ *given in (9) of Theorem 2.9.*

**Remark:** The part (b) of Corollary 3.3 describes the behaviour of the average statistical information measure when one observes $X$ only in the time interval $[u, (u+\Delta t)\wedge T]$ (rather than $[u,T]$), where the (maximum) interval size $\Delta t$ tends to zero. In contrast, the Part (c) estimates the time-evolution of the average statistical information measure for *any* interval size $\Delta t$.

**Corollary 3.4** *Suppose the Assumptions 2.7 and 2.8. Then the following assertions hold :*
*(a) for all* $\alpha \in\, ]0,1[$ *and all prior probabilities* $p \in [0,1[$ *:*

$$\lim_{\Delta t \downarrow 0} \sup_{(u,x)\in[0,T]\times]0,\infty[} \Delta\mathcal{BR}_{\alpha,p}\Big(p, Q_{(u,x)}^{(u+\Delta t)\wedge T}, P_{(u,x)}^{(u+\Delta t)\wedge T}\Big) \;=\; 0 \quad .$$

*(b) according to the value of* $\alpha \in ]0,1[$*, for any fixed starting time* $u \in [0,T]$*, any starting point* $x > 0$*, and any prior probability* $p$*, the time evolution (with respect to* $\Delta t > 0$*) of the statistical information measure*

$$\Delta\mathcal{BR}_{\alpha,p}\Big(p, Q_{(u,x)}^{(u+\Delta t)\wedge T}, P_{(u,x)}^{(u+\Delta t)\wedge T}\Big)$$

*can be estimated from above by the function* $h_3(\Delta t)$ *given in (9) of Theorem 2.9. In particular, these estimates do not depend on the prior probability* $p$*.*

**Remark:** For the special case of supposing the Assumption 2.7 together with the non-stochastic condition (7) in the two Corollaries 3.3 and 3.4, see also Stummer [26].

## 4 Applications to Equivalent Martingale Measures and Contingent Claim Pricing

As already mentioned above, the asset price model $(X_t, Q_{(s,\mu)})$ described by the SDE (2) can be interpreted as a non-lognormally-distributed perturbation of Samuelson's geometric Brownian motion described by the SDE (1). The latter was used by Black & Scholes [1] and Merton [19] to describe the dynamic behaviour of the underlying in their investigations on option pricing; later on, the more general pricing framework of equivalent martingale measures was introduced by Harrison & Kreps [7], and Harrison & Pliska [8],[9].

In this section, we apply some of the results of Section 2 in order to derive (i) the existence of a (unique) equivalent martingale measure by a "direct" method, as well as the corresponding (ii) no-arbitrage results, and (iii) option pricing assertions, for a financial market $\mathcal{M} = \{B, X\}$, which consists of
(a) a bond $B$, whose price evolution is given by $B_t = e^{\int_0^t r_v dv}$, where the deterministic short rate process $r_t$ is nonnegative and continuous in $t$,
(b) a financial asset $X$, whose price process is a weak solution of the SDE (2) with time-**homogeneous** drift $b(X_v)$. Also, the financial asset $X$ is assumed to continuously pay dividends of the amount $\delta_t X_t dt$ between time $t$ and $t + dt$, where the dividend yield $\delta_t$ is a deterministic, continuous, nonnegative function of $t$.

Furthermore, we use the standard assumptions that the lending (interest) rate is equal to the borrowing (interest) rate, that there are no transaction costs and no taxes, and that trading takes place continuously.

As usual, we employ the discounted price process $\widetilde{X}_t := X_t \; e^{\int_0^t (\delta_v - r_v) dv}$. Here, this amounts to studying the discounted SDE

$$d\widetilde{X}_t \;=\; \widetilde{b}(t, \widetilde{X}_t)\; dt \;+\; \sigma_t \; \widetilde{X}_t \; d\widetilde{W}_t \;, \quad t \in [s, T], \tag{13}$$

$$\widetilde{X}_s \textit{ has probability distribution } \widetilde{\mu} \textit{ on } ]0, \infty[ \;,$$

with discounted drift

$$\widetilde{b}(t, y) := (\delta_t - r_t) y \;+\; e^{\int_0^t (\delta_v - r_v) dv} \;\; b\Big(e^{\int_0^t (r_v - \delta_v) dv} y\Big)$$

(notice that $\widetilde{b}$ is – in contrast to $b$ – time-inhomogeneous). The corresponding solution measure will be denoted by $\widetilde{Q_{(s,\widetilde{\mu})}}$; we also use the notations $\widetilde{P_{(s,\widetilde{\mu})}}$ and $\widetilde{\widetilde{W}}$ for the case $\widetilde{b}(t, y) \equiv 0$.

Before we start with our investigations, let us first notice notice that – even in the case of zero interest rates and zero dividend yields – there is no automatism: the fact that the asset price process $X$ is modelled by a stochastic differential equation does generally not imply the existence of an equivalent martingale measure, no-arbitrage, and so on. As a counterexample, one can take the Bessel process with dimension three (see Delbaen & Schachermayer [4]). Thus, one needs some further considerations.

For our "generalized Samuelson-Merton-Black-Scholes world", we proceed as follows:

**Theorem 4.1** *Suppose the Assumptions 2.7 and 2.8. Then all the assertions (a) to (c) of Theorem 2.9, (a) to (c) of Corollary 3.3, and (a) to (b) of Corollary 3.4 also hold for the* **discounted** *asset price model (i.e. with $\widetilde{Q}$ instead of $Q$, $\widetilde{P}$ instead of $P$, $\widetilde{b}$ instead of $b$, $\widetilde{\mu}$ instead of $\mu$, $\widetilde{x}$ instead of $x$, and so on).*

**Remark:** Whereas the assumptions are made on the undiscounted world, the assertions concern the discounted world.

With the help of (the first part of) the above Theorem 4.1, one obtains the existence of a unique equivalent martingale measure by application of the Girsanov theorem in the opposite direction:

**Theorem 4.2** *Suppose the Assumptions 2.7 and 2.8. Then one has for all starting times $s \in [0, T]$ and all starting distributions $\widetilde{\mu}$ on $]0, \infty[$:*
*(a) the Girsanov property*

$$E\widetilde{Q_{(s,\widetilde{\mu})}} \left[ \widetilde{Z^{opp}_{s,T}} \right] = 1 \tag{14}$$

*holds, where*

$$\widetilde{Z^{opp}_{s,t}} := \exp\left( - \int_s^t \frac{\widetilde{b}(v, \widetilde{X_v})}{\sigma_v \widetilde{X_v}} \, d\widetilde{W}_v - \frac{1}{2} \int_s^t \frac{\widetilde{b}(v, \widetilde{X_v})^2}{\sigma_v^2 \, (\widetilde{X_v})^2} \, dv \right), \qquad t \in [s, T] \, .$$

*(b) $\widetilde{P_{(s,\widetilde{\mu})}}$ is an equivalent martingale measure for $(\widetilde{X}, \widetilde{W}, \widetilde{Q_{(s,\widetilde{\mu})}})$.*
*(c) within the class of weak solutions $(\widetilde{\widetilde{X}}, \widetilde{\widetilde{W}}), \widetilde{\widetilde{Q_{(s,\widetilde{\mu})}}})$ of (13) with the additional property*

$$\widetilde{\widetilde{Q_{(s,\widetilde{\mu})}}} \left[ \int_s^T \frac{\widetilde{b}(v, \widetilde{\widetilde{X_v}})^2}{\sigma_v^2 \, \widetilde{\widetilde{X_v}}^2} \, dv < \infty \right] = 1,$$

*the equivalent martingale measure $\widetilde{P_{(s,\widetilde{\mu})}}$ is unique.*

**Remark:** We have called the condition (14) "Girsanov **property**", for the sake of a better verbal distinguishability with its opposite counterpart, the (appropriately adapted version of the) Girsanov condition (5).

Next, we apply the above result in order to investigate the absence of arbitrage opportunities for the market $\mathcal{M} = \{B, X\}$. For this, and the rest of this section, we fix $s = 0$ as the starting (trading) time. Let us recall the following standard terminology: a **trading strategy** $\phi := \{\phi^{(0)}, \phi^{(1)}\}$ in the market $\mathcal{M}$ consists of (i) a $I\!R-$valued, progressively measurable process $\phi_t^{(0)}$, which denotes the number of shares of the bond $B$ which are held at time $t$, and (ii) a $I\!R-$valued, progressively measurable stochastic process $\phi_t^{(1)}$ which denotes the number of shares of the financial asset $X$ which are held at time $t$. The **portfolio value** at time $t$, which corresponds to the trading strategy $\phi$, is given by $\mathcal{PO}_t := \phi_t^{(0)}\ B_t + \phi_t^{(1)} X_t$. The **gains process** associated with the trading strategy $\phi$ is defined, under $Q_{(0,x)}$, as,

$$G_t \ = \ \int_0^t \phi_v^{(0)}\ r_v\ B_v\ dv \ + \ \int_0^t \phi_v^{(1)}\ dX_v \ + \ \int_0^t \phi_v^{(1)}\ \delta_v\ X_v\ dv$$

where the integrals should exist. A trading strategy $\phi$ is called **self-financing**, if the corresponding portfolio process satisfies $\mathcal{PO}_t \ = \ \mathcal{PO}_0 \ + \ G_t$ for all $t \in [0,T]$.

**Definition 4.3** *A self-financing trading strategy $\phi$ is called an* **arbitrage opportunity** *if the corresponding portfolio process satisfies the three conditions*

$$(i) \quad \mathcal{PO}_0 = 0 \ ,$$

$$(ii) \quad Q_{(0,x)}\Big[\ \mathcal{PO}_T \geq 0\ \Big] \ = \ 1 \ ,$$

$$(iii) \quad Q_{(0,x)}\Big[\ \mathcal{PO}_T > 0\ \Big] \ > \ 0 \ .$$

With these notations, one obtains the following arbitrage theorem:

**Theorem 4.4** *Suppose the Assumptions 2.7 and 2.8. Then, for all starting prices $x \in\ ]0,\infty[$ at time 0, there are no arbitrage opportunities in the class of all self-financing trading strategies $\phi := \{\phi^{(0)}, \phi^{(1)}\}$ which satisfy the three assumptions*

$$(i) \quad Q_{(0,x)}\Big[\ \int_0^T \left|\phi_v^{(0)}\right|\ dv < \infty\ \Big] \ = \ 1 \ ,$$

$$(ii) \quad Q_{(0,x)}\Big[\ \int_0^T \left(\sigma_v\ \phi_v^{(1)}\right)^2\ dv < \infty\ \Big] \ = \ 1 \ ,$$

*(iii) the (discounted gains) process $\widetilde{G_t} := \int_0^t \frac{\phi_v^{(1)}\sigma_v X_v}{B_v} d\widetilde{\widetilde{W}}_v$ is a $\widetilde{P_{(0,x)}}$-supermartingale.*

Finally, as an application of the above results, let us now provide the corresponding valuation theorem of European contingent claims on an underlying asset $X$ whose price processes are the here-treated non-lognormally distributed generalizations of the geometric Brownian motion.

Recall the standard terminology that a **European (type) contingent claim** $\mathcal{H}$ **with expiration date T** is, from a mathematical point of view, nothing but a $\mathcal{F}_T-$measureable random variable, where $\mathcal{F}_T$ is the largest sigma-algebra from the (up to here omitted) corresponding solution filtration $(\mathcal{F}_t)$. The most prominent example is the **European call option** $\mathcal{H} = \max(X_T - \mathcal{K}, 0)$ **with** deterministic **strike price** $\mathcal{K} > 0$.

**Theorem 4.5** *Suppose the Assumptions 2.7 and 2.8. Then, for all starting prices* $x \in\ ]0, \infty[$, *one gets the following statements:*
*(a) For every European contingent claim* $\mathcal{H} \in L^1(\widetilde{P_{(0,x)}})$ *with expiration date* $T$, *the unique arbitrage-based price* $V_t$ *at time* $t \in [0,T]$ *is given by the formula* ($Q_{(0,x)}-$*almost surely)*

$$V_t \ = \ e^{-\int_t^T r_v dv} \ E\widetilde{P_{(0,x)}}[\ \mathcal{H} \mid \mathcal{F}_t] \ = \ e^{-\int_t^T r_v dv} \ E\widetilde{P_{(t,\widetilde{X}_t)}}[\ \mathcal{H}\ ] \ . \tag{15}$$

*(b) In particular, the unique arbitrage-based price* $V_t^{CALL}$ *of a European call option on the underlying asset* $X$ *with strike price* $\mathcal{K} > 0$ *and expiration date* $T$ *is given by*

$$V_t^{CALL} \ = \ X_t \ e^{-\int_t^T \delta_v dv} \ \ F_N(d_1) \ - \ \mathcal{K} \ e^{-\int_t^T r_v dv} \ \ F_N(d_2) \ , \tag{16}$$

$Q_{(0,x)}-$*almost surely, with*

$$d_1 \ := \ \frac{log\left(\frac{X_t}{\mathcal{K}}\right) \ + \ \int_t^T \left(r_v - \delta_v + \frac{\sigma_v^2}{2}\ \right) \ dv}{\sqrt{\int_t^T \sigma_v^2 \ dv}} \quad and \quad d_2 \ := \ d_1 \ - \ \sqrt{\int_t^T \sigma_v^2 \ dv} \ .$$

**Remarks:** (R1) The original Black-Scholes theorem [1] can be derived as a special case of the part (b) of Theorem 4.5, by taking the linear drift and constant volatility of the geometric Brownian motion SDE (1) together with constant short rate $r_t \equiv r^{co}$ and zero dividend yield $\delta_t \equiv 0$. (The non-stochastic interest-rate version of) Merton's theorem [19] deals with the same SDE set-up (1), but with non-constant short rates $r_t$ and constant dividend yield $\delta_t \equiv \delta^{co}$; Rubinstein [23] uses non-constant dividend yields $\delta_t$. Those cases are also covered by Theorem 4.5.
(R2) In the context of "real options" one is sometimes using the Black-Scholes or Merton's formula, although one knows that the underlying quantity can only be approximated by a geometric Brownian motion; see e.g. Kemna [11] and Carr [2]. As a tool to support such an action plan, the

non-stochastic condition (7) involved in Theorem 4.5 delivers a handy-to-verify, non-stochastic toolbox for obtaining a variety of non-lognormally distributed underlyings $X$, such that one can still valuate the corresponding call options with the Black-Scholes formula or Merton's formula.
(R3) The valuation formulae (15) and (16) do not depend on the modelling drift $b$. However, the knowledge of $b$ is important for the implementation of, say, the formula (16), if for the input $\sigma_t$ one uses a discretely sampled historical estimate. For a general discussion on this subject, the reader is referred to Lo & Wang [14]. Furthermore, although the formula (16) is drift-less, it can not be taken for granted to be valid for all underlyings $X$ with arbitrary drifts $b$. For very exploding drifts $b$ there may not even exist an equivalent martingale measure, and thus the formula (16) is not valid anymore. Hence, it is important to have assumptions on $b$.
(R4) For the special case of supposing the Assumption 2.7 together with the non-stochastic condition (7) in the Theorems 4.1 to 4.5, see also Stummer [27], where one can find examples, too.

## 5 Proofs

**Proof of Theorem 2.3.** Under the Assumption 2.1, all the 6 conditions (a) to (f) in the Assumption 2.2 are equivalent. This can be deduced as a special case of the general considerations upon Markov processes in Stummer & Sturm [28]. Consequently, the assertions (a) and (b) of Theorem 2.3 follow then from the well-known theorems of Novikov [21] and Girsanov [6] (for the latter, see also Maruyama [15],[16]); the positivity property is inherited from the drift–less case. The assertion (c) can be proved by applying (an adapted version of) the uniqueness result of Karatzas & Shreve [10], p.304 (see also Rydberg [24]). □

**Proof of Theorem 2.4.** Firstly, under the Assumptions 2.1 and 2.2, the set $\mathcal{U}$ is always non–empty. This follows again from the abovementioned general considerations upon Markov processes in Stummer & Sturm [28]. Furthermore, it suffices to prove the part (b). The lower bound 0 for $I_\alpha$ follows immediately from the positiveness of the function $f_\alpha$. For the proof of the upper bounds, we distinguish:

<u>**Case 1.** $\alpha = \mathbf{0}$ :</u> For fixed time $u \in [0,T]$ and fixed $x \in ]0,\infty[$ one has, because of the Girsanov condition (5) and the martingale property of the appearing stochastic integral,

$$I_\alpha(Q_{(u,x)}||P_{(u,x)}) \;=\; EP_{(u,x)}\Big[\,\frac{dQ_{(u,x)}}{dP_{(u,x)}} \;-\; 1 \;-\; \log\Big(\frac{dQ_{(u,x)}}{dP_{(u,x)}}\Big)\,\Big]$$

$$= \;-\; EP_{(u,x)}\Big[\,\log\Big(\frac{dQ_{(u,x)}}{dP_{(u,x)}}\Big)\,\Big]$$

$$= \; EP_{(u,x)}\Big[ \; - \int_u^T \frac{b(v,X_v)}{\sigma_v \; X_v} \; d\overline{W}_v \;\; + \; \frac{1}{2} \; \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2 \; (X_v)^2} \; dv \Big]$$

$$= \; EP_{(u,x)}\Big[ \; \frac{1}{2} \; \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2 \; (X_v)^2} \; dv \; \Big] \; .$$

<u>**Case 2.** $\alpha \; \in \mathbf{R}\backslash[\mathbf{0},\mathbf{1}]$ :</u> Again with the help of (5), one can calculate for fixed time $u \in [0,T]$ and fixed $x \in ]0,\infty[$

$$I_\alpha(Q_{(u,x)}||P_{(u,x)}) = \frac{1}{\alpha \; (1-\alpha)} \; EP_{(u,x)}\Big[\alpha \; \frac{dQ_{(u,x)}}{dP_{(u,x)}} + 1 - \alpha - \big(\frac{dQ_{(u,x)}}{dP_{(u,x)}}\big)^\alpha\Big]$$

$$= \; \frac{1}{\alpha \; (1-\alpha)} \; + \; \frac{1}{\alpha \; (\alpha-1)} \; EP_{(u,x)}\Big[ \; \big(\frac{dQ_{(u,x)}}{dP_{(u,x)}}\big)^\alpha \; \Big]$$

$$= \; \frac{1}{\alpha \; (1-\alpha)} \; + \; \frac{1}{\alpha \; (\alpha-1)} \;\; EP_{(u,x)}\Big[exp\Big(\alpha \int_u^T \frac{b(v,X_v)}{\sigma_v \; X_v} \; d\overline{W}_v$$

$$- \; \frac{\alpha}{2} \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2 \; (X_v)^2} \; dv\Big) \; \Big]$$

$$= \frac{1}{\alpha \; (1-\alpha)} + \frac{1}{\alpha \; (\alpha-1)} \; EP_{(u,x)}\Big[exp\Big( \; \frac{\alpha^2 q_2 - \alpha}{2} \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2 \; (X_v)^2} \; dv\Big) \; \times$$

$$\times \; exp\Big(\alpha \; \int_u^T \frac{b(v,X_v)}{\sigma_v \; X_v} \; d\overline{W}_v \; - \; \frac{\alpha^2 q_2}{2} \; \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2 \; (X_v)^2} \; dv\Big) \; \Big]$$

$$\leq \; \frac{1}{\alpha \; (1-\alpha)} \; + \; \frac{1}{\alpha \; (\alpha-1)} \; \Big\{ \; EP_{(u,x)}\Big[ \; exp\Big( \; \frac{\alpha^2 (q_2)^2 - \alpha q_2}{2 \; (q_2-1)}$$

$$\times \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2 \; (X_v)^2} \; dv\Big)\Big]\Big\}^{\frac{q_2-1}{q_2}} \; \Big\{EP_{(u,x)}\Big[ \; exp\Big(\alpha q_2 \int_u^T \frac{b(v,X_v)}{\sigma_v \; X_v} \; d\overline{W}_v$$

$$- \; \frac{\alpha^2 (q_2)^2}{2} \; \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2 \; (X_v)^2} \; dv\Big) \; \Big]\Big\}^{\frac{1}{q_2}} \; , \tag{17}$$

where Hölder's inequality has been used with any $q_2 > 1$. It is not clear that the first expectation in (17) is finite, even for fixed time $u \in [0,T]$ and fixed $x$ (and we even want the supremum). In order to investigate if — and for which $q_2$ — this might be the case, let us fix any arbitrary $\epsilon > 0$

for which the condition

$$\sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)}\Big[\ exp\Big(\frac{1+\epsilon}{2}\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\ (X_v)^2}\ dv\Big)\Big]\ <\ \infty$$

holds. It thus suffices to find an exponent $q_2 > 1$ such that

$$\Upsilon_1(q_2)\ :=\ \frac{\alpha^2(q_2)^2 - \alpha q_2}{2\ (q_2-1)}\ \leq\ \frac{1+\epsilon}{2}\ . \tag{18}$$

Let us therefore study the function $\Upsilon_1$ closer; recall that we are in the case $\alpha\ \in I\!R\backslash[0,1]$. It is easy to see that $q_2 \to \Upsilon_1(q_2)$ is a strictly convex function on $]1,\infty[$ attaining a minimum at the point $q_2^{min} := q_2^{min}(\alpha) := 1+\sqrt{1-\frac{1}{\alpha}}$. Also, one gets $\Upsilon_1(q_2^{min}(\alpha)) = \frac{\alpha^2\ (q_2^{min}(\alpha))^2}{2}$. In order to find the corresponding largest possible range of admissible parameters $\alpha$, one has to solve the equation

$$\frac{\alpha^2\Big(1+\sqrt{1-\frac{1}{\alpha}}\ \Big)^2}{2}\ =\ \frac{1+\epsilon}{2}\ . \tag{19}$$

With the definition $\Upsilon_2 := \sqrt{1-\frac{1}{\alpha}}$, the equation (19) can be transformed to the equation $|1-\Upsilon_2| = (1+\epsilon)^{-1/2}$. By retransforming, one gets hence

$$\alpha^{min} = \frac{1}{1-(1+\frac{1}{\sqrt{1+\epsilon}})^2} \qquad\qquad for\ the\ subcase\ \alpha < 0\ ,$$

and

$$\alpha^{max} = \frac{1}{1-(1-\frac{1}{\sqrt{1+\epsilon}})^2} \qquad\qquad for\ the\ subcase\ \alpha > 1\ .$$

If one chooses $\tilde{\alpha} \in [\alpha^{min},\alpha^{max}]\backslash[0,1] = [h_1(\epsilon),0[\ \cup\ ]1,h_2(\epsilon)]$, and $\tilde{q}_2 = q_2^{min}(\tilde{\alpha})$, then one obtains

$$\sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)}\Big[\ exp\Big(\frac{\tilde{\alpha}^2(\tilde{q}_2)^2}{2}\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\ (X_v)^2}\ dv\Big)\Big]\ <\ \infty\ . \tag{20}$$

Thus, for this choice, the first expectation in (17) is bounded with respect to $u \in [0,T]$ and $x \in ]0,\infty[$. Also, condition (20) expresses that the modified drift "$\tilde{\alpha}\ \tilde{q}_2\ b$" satisfies the uniform Novikov condition and hence, the Girsanov condition (5) holds also for this drift "$\tilde{\alpha}\ \tilde{q}_2\ b$" and any starting constellation $(u,x)$. Therefore, the second expectation in (17) is equal to 1 for all $u \in [0,T]$ and $x \in ]0,\infty[$. This concludes the proof of Case 2.

**Case 3.** $\alpha \in ]0,1[$ **:** As in Case 2, one can calculate for any starting time $u \in [0,T]$, any starting point $x \in ]0,\infty[$, and any $q_2 \in ]0,1[$ (rather than $q_2 > 1$)

$$
\begin{aligned}
& I_\alpha(Q_{(u,x)}||P_{(u,x)}) \\
&= \frac{1}{\alpha\,(1-\alpha)} - \frac{1}{\alpha\,(1-\alpha)}\; EP_{(u,x)}\Big[ exp\Big( \frac{\alpha^2 q_2 - \alpha}{2} \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv \Big) \\
&\times exp\Big( \alpha \int_u^T \frac{b(v,X_v)}{\sigma_v\,X_v}\,d\overline{W}_v \;-\; \frac{\alpha^2 q_2}{2} \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv \Big) \Big] \\
&\leq \frac{1}{\alpha\,(1-\alpha)} + \frac{1}{\alpha\,(\alpha-1)} \;\Big\{ EP_{(u,x)}\Big[ exp\Big( \frac{\alpha^2 (q_2)^2 - \alpha q_2}{2\,(q_2-1)} \\
&\times \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv \Big) \Big] \Big\}^{\frac{q_2-1}{q_2}} \;\Big\{ EP_{(u,x)}\Big[ exp\Big( \alpha q_2 \int_u^T \frac{b(v,X_v)}{\sigma_v\,X_v}\,d\overline{W}_v \\
&- \frac{\alpha^2 (q_2)^2}{2} \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv \Big) \Big] \Big\}^{\frac{1}{q_2}} ,
\end{aligned} \tag{21}
$$

where Hölder's inequality has been used for $q_2 \in ]0,1[$. Let us remark that — in contrast to Case 2 — the factor $\frac{1}{\alpha\,(\alpha-1)}$ in front of the expectation is now negative, and thus, the application of Hölder's inequality with $q > 1$ would yield the $\geq$ symbol in (21). Since Hölder's inequality with $q \in ]0,1[$ looks just like Hölder's inequality with $q > 1$ but with reverted inequalities, the choice $q \in ]0,1[$ produces the wanted $\leq$ symbol in (21).

The second expectation in (21) is equal to 1 since, for any $q_2 \in ]0,1[$, the modified drift term "$\alpha q_2 b$" also satisfies the uniform Novikov condition; hence the appropriate version of the Girsanov condition (5) can be applied. It is easy to see that, for fixed $\alpha \in ]0,1[$, the function $q_2 \to \Upsilon_1(q_2)$ (defined in (18)) is on the interval $]0,1[$ a strictly increasing, strictly convex function with $\lim_{q_2 \downarrow 0} = 0$ and $\lim_{q_2 \uparrow 1} = \infty$. Thus, with the notation of Case 2, one can always choose a unique exponent $\tilde{q_2} \in ]0,1[$ for which $\Upsilon_1(\tilde{q_2}) = \frac{1}{2}$. Namely, by solving this quadratic equation one gets

$$
\tilde{q_2} \;=\; \frac{\alpha - 1 + \sqrt{1 - 2\alpha + 5\alpha^2}}{2\alpha^2} \;. \tag{22}
$$

Thus, one can compute

$$
\frac{1-\tilde{q_2}}{\tilde{q_2}} \;=\; \alpha - \alpha^2 \tilde{q_2} \;=\; \frac{\alpha + 1 - \sqrt{1 - 2\alpha + 5\alpha^2}}{2} \;, \tag{23}
$$

which also shows that $\tilde{q_2}$ is indeed smaller than 1. Hence, we have finished the proof of Case 3.

**Case 4. $\alpha = 1$ :** For fixed time $u \in [0,T]$ and fixed $x \in ]0,\infty[$, if under the assumption (4) one can show

$$EQ_{(u,x)}\Big[\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big] < \infty\,, \tag{24}$$

then one can calculate, because of (5) and the martingale property of the appearing stochastic integral,

$$\begin{aligned}
I_\alpha(Q_{(u,x)}||P_{(u,x)}) &= EP_{(u,x)}\Big[1 - \frac{dQ_{(u,x)}}{dP_{(u,x)}} + \frac{dQ_{(u,x)}}{dP_{(u,x)}}\log\Big(\frac{dQ_{(u,x)}}{dP_{(u,x)}}\Big)\Big]\\
&= EP_{(u,x)}\Big[\frac{dQ_{(u,x)}}{dP_{(u,x)}}\log\Big(\frac{dQ_{(u,x)}}{dP_{(u,x)}}\Big)\Big]\\
&= EQ_{(u,x)}\Big[\int_u^T \frac{b(v,X_v)}{\sigma_v\,X_v}\,d\overline{W}_v - \frac{1}{2}\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big]\\
&= EQ_{(u,x)}\Big[\int_u^T \frac{b(v,X_v)}{\sigma_v\,X_v}\Big(\frac{b(v,X_v)}{\sigma_v\,X_v}\,dv + dW_v\Big) - \frac{1}{2}\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big]\\
&= EQ_{(u,x)}\Big[\int_u^T \frac{b(v,X_v)}{\sigma_v\,X_v}\,dW_v\Big] + \frac{1}{2}EQ_{(u,x)}\Big[\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big]\\
&= \frac{1}{2}EQ_{(u,x)}\Big[\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big]\,.
\end{aligned}$$

Thus, we have to show the finiteness of the last term (as e.g. in Föllmer [5]). For any $q_1 \in ]1, \sup_{\epsilon\in\mathcal{U}} h_2(\epsilon)[$, let us therefore further estimate via Hölder's inequality

$$\begin{aligned}
I_\alpha(Q_{(u,x)}||P_{(u,x)}) &= \frac{1}{2}EP_{(u,x)}\Big[\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\\
&\times exp\Big(\int_u^T \frac{b(v,X_v)}{\sigma_v\,X_v}\,d\overline{W}_v - \frac{1}{2}\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big)\Big]\\
&\le \frac{1}{2}\Big\{EP_{(u,x)}\Big[\Big(\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big)^{\frac{q_1}{q_1-1}}\Big]\Big\}^{\frac{q_1-1}{q_1}}\\
&\times\Big\{EP_{(u,x)}\Big[exp\Big(q_1\int_u^T \frac{b(v,X_v)}{\sigma_v\,X_v}\,d\overline{W}_v - \frac{q_1}{2}\int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\,(X_v)^2}\,dv\Big)\Big]\Big\}^{\frac{1}{q_1}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2} \sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)}\Big[ \Big( \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\ (X_v)^2}\ dv \Big)^{\frac{q_1}{q_1-1}} \Big]^{\frac{q_1-1}{q_1}} \\
&\times \sup_{(u,x)\in[0,T]\times]0,\infty[} \Big\{ EP_{(u,x)}\Big[ exp\Big( \frac{(q_1)^2(1+\sqrt{1-\frac{1}{q_1}})^2}{2} \\
&\times \int_u^T \frac{b(v,X_v)^2}{\sigma_v^2\ (X_v)^2}\ dv \Big) \Big]^{\frac{\sqrt{1-\frac{1}{q_1}}}{1+\sqrt{1-\frac{1}{q_1}}}} \Big\}^{\frac{1}{q_1}} ,
\end{aligned}
$$

where we have proceeded as in Case 2 above (with $q_1$ in place of $\alpha$). Both expectations after the last inequality are finite because of (20) (with $\tilde{\alpha} := q_1$), which concludes the proof of Case 4. □

**Proof of Theorem 2.6.** The assertions follow in a straightforward manner from a closer look at the proof of Theorem 2.4. □

**Proof of Theorem 2.9.** As already mentioned above, under the condition (3) all the 6 conditions (a) to (f) in the Assumption 2.2 are equivalent. Furthermore, one can show that under the two conditions (3) and (6) also the non-stochastic condition (7) is equivalent to these 6 conditions; in other words, all the 7 conditions mentioned in the Assumption 2.8 are equivalent.

To see this, let us first mention that in Stummer [26] it is shown that the condition (7) implies the condition

$$
\lim_{\Delta t \downarrow 0} \sup_{(u,x)\in[0,T]\times]0,\infty[} EP_{(u,x)} \Big[ \int_u^{(u+\Delta t)\wedge T} \frac{b(X_v)^2}{\sigma_v^2\ (X_v)^2}\ dv \Big] = 0 .
$$

From this, (the time-homogeneous-drift version of) the uniform Novikov condition (4) follows by applying a generalized version of the Khas'minskii-Lemma (see e.g. Stummer & Sturm [28]). The reverse implication (4) $\Longrightarrow$ (7) follows as in Stummer [26], where one can also find the deduction of the assertions (a) to (c) of Theorem 2.9 from the condition (7). □

**Proof of Theorem 4.1.** Since, under the Assumption 2.7, all the 7 conditions in Assumption 2.8 are equivalent, one can start with the non-stochastic condition (7). But (3), (6) and (7) imply the condition

$$
\lim_{\Delta t \downarrow 0} \sup_{(u,\widetilde{x})\in[0,T]\times]0,\infty[} E\widetilde{P_{(u,\widetilde{x})}} \Big[ \int_u^{(u+\Delta t)\wedge T} \frac{\widetilde{b}(v,\widetilde{X}_v)^2}{\sigma_v^2\ (\widetilde{X}_v)^2}\ dv \Big] = 0 ; \tag{25}
$$

from this, one can proceed similarly to the proof of Theorem 2.9 (for details, the reader is referred to Stummer [27]). The rest follows by using the appropriately adapted versions of the representation formulae (10) and (12). □

**Proof of Theorem 4.2.** It is straightforward to see that, for the case $\alpha = 1$, the condition (8) in Theorem 2.9 – appropriately adapted to the discounted world – is equivalent to the condition

$$\lim_{\Delta t \downarrow 0} \sup_{(u,\widetilde{x}) \in [0,T]\times ]0,\infty[} E\widetilde{Q_{(u,\widetilde{x})}} \left[ \int_u^{(u+\Delta t)\wedge T} \frac{\widetilde{b}(v,\widetilde{X}_v)^2}{\sigma_v^2 \ (\widetilde{X}_v)^2} \ dv \right] = 0 \ .$$

Thus, by applying again the abovementioned generalized version of the Khas'minskii-Lemma in Stummer & Sturm [28], one gets the "uniform Novikov property"

$$\sup_{(u,\widetilde{x}) \in [0,T]\times ]0,\infty[} E\widetilde{Q_{(u,\widetilde{x})}} \Big[ \ exp\Big(\frac{1}{2} \int_u^T \frac{\widetilde{b}(v,\widetilde{X}_v)^2}{\sigma_v^2 \ (\widetilde{X}_v)^2} \ dv\Big)\Big] < \ \infty \ .$$

Hence, from the Novikov Theorem one deduces the part (a). From this and the Girsanov theorem, one obtains that the measure $\widetilde{P_{(s,\widetilde{\mu})}}$ is absolutely continuous with respect to (and hence, equivalent to) the solution measure $\widetilde{Q_{(s,\widetilde{\mu})}}$. Since the drift-less discounted price process $(\widetilde{X}, \widetilde{P_{(s,\widetilde{\mu})}})$ is a martingale, the part (b) holds. Because of (25) and the appropriately adapted version of part (c) of Theorem 2.3, the part (c) of Theorem 4.2 can be obtained e.g. by using the uniqueness result of Rydberg [24]. □

**Proof of Theorem 4.4.** By straightforward calculations it is easy to see that under (i) and (ii), all the involved quantities in the corresponding gains process $G_t$ do ($Q_{(0,x)}-$ almost surely) exist. The rest can be deduced by standard techniques, with the help of the characterization formula $\frac{\mathcal{PO}_t}{B_t} - \mathcal{PO}_0 = \widetilde{G}_t$. □

**Proof of Theorem 4.5.** With the help of standard techniques, the assertions follow in a straightforward manner from the above considerations and the martingale representation theorem. □

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